ABSTRACT FIRST ORDER COMPUTABILITY. II[1]

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In *Abstract first order computability*. I (the preceding paper), we initiated a study of computability on abstract structures and carried it up to the theory of the "projective hierarchy," our generalization of the arithmetical hierarchy on the set of integers. Here we study the "hyperprotective hierarchy" which is our abstract version of the hyperarithmetic hierarchy on the integers. The numbering of paragraphs and results is a continuation of the numbering in the first part and we have collected in a partial bibliography at the end those papers which are referred to in this part.

10. **Hyperprojective functions.** In XLIII of [6] Kleene characterizes the hyperarithmetic number-theoretic functions as exactly those functions which are recursive in the type-2 object ${}^2\mathbf{E}$ which embodies number quantification. Similarly here we wish to study the class of functions which are search computable in the object $\mathbf{E}_{B^*}=\mathbf{E}$ which embodies quantification over B^*. We define $\mathbf{E}(\mathrm{g})$ for a one-place p.m.v. function g by

$$(10.1)\quad \begin{aligned} \mathbf{E}(\mathrm{g}) \to z \Leftrightarrow {} & [z = 0 \,\&\, (y)\mathrm{g}(y)\downarrow \,\&\, (Ey)\mathrm{g}(y) \to 0] \\ & \vee [z = 1 \,\&\, (y)(Eu)[u \neq 0 \,\&\, \mathrm{g}(y) \to u]]; \end{aligned}$$

it is clear that for single-valued total g we have

$$(10.2)\quad \begin{aligned} \mathbf{E}(\mathrm{g}) &= 0 \quad \text{if } (Ey)[\mathrm{g}(y) \to 0], \\ &= 1 \quad \text{otherwise.} \end{aligned}$$

We define the class $\mathbf{HP}(\boldsymbol{\varphi})$ of functions *computable* in $\mathbf{E}$, $\boldsymbol{\varphi}$ or *hyperprojective* (*in* $\boldsymbol{\varphi}$) by adding to C0–C9 the schema

C10. $\mathrm{f}(\boldsymbol{x})=\mathbf{E}(\lambda y\mathrm{g}(y, \boldsymbol{x}))\quad \langle 10, n, g\rangle$,

and accordingly adding to C0′–C9′ the clause

C10′. *If* $(y)(Eu)[\{g\}(y, \boldsymbol{x}) \to u] \,\&\, (Ey)[\{g\}(y, \boldsymbol{x}) \to 0]$, *then* $\{\langle 10, n, g\rangle\}(\boldsymbol{x}) \to 0$; *if* $(y)(Eu)[u\neq 0 \,\&\, \{g\}(y, \boldsymbol{x}) \to u]$, *then* $\{\langle 10, n, g\rangle\}(\boldsymbol{x}) \to 1$.

In the usual way clauses C0′–C10′ define a predicate $\{f\}_{\mathrm{h}}(\boldsymbol{u}) \to z$ and assign to each $f \in B^*$ a p.m.v. function $\{f\}_{\mathrm{h}}(\boldsymbol{u})$. We let $HP(A, \boldsymbol{\varphi})=HP(A)$ be the class of functions *hyperprojective from* A, i.e. functions $\{f\}_{\mathrm{h}}(\boldsymbol{u})$ with $f \in A^*$ and put $HP(\boldsymbol{\varphi}) = HP = HP(\varnothing)$ (the *absolutely hyperprojective* functions), $\mathbf{HP}(\boldsymbol{\varphi})=\mathbf{HP}=HP(B)$, the hyperprojective functions. For each list of variables $\mathrm{g}_1, \ldots, \mathrm{g}_m$ we get a functional $\{f\}_{\mathrm{h}}(\mathrm{g}_1, \ldots, \mathrm{g}_m, \boldsymbol{u})$.

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In order to assign an ordinal $|f, \boldsymbol{u}, z|_{\mathrm{h}}$ to each sequence $f, \boldsymbol{u}, z$ such that $\{f\}_{\mathrm{h}}(\boldsymbol{u}) \to z$, we add to C0″–C9″ clause

C10″. *If* $(y)(Eu)[\{g\}(y, \boldsymbol{x}) \to u]$ & $(Ey)[\{g\}(y, \boldsymbol{x}) \to 0]$, *then*

$$|\langle 10, n, g\rangle, \boldsymbol{x}, 0| = \mathit{infimum}\{\mathit{supremum}[|g, y, \boldsymbol{x}, \alpha(y)|+1 : \mathit{all}\ y] \\ :(y)[\{g\}(y, \boldsymbol{x}) \to \alpha(y)] \;\&\; (Ey)\alpha(y) \neq 0\};$$

if $(y)(Eu)[\{g\}(y, \boldsymbol{x}) \to u \;\&\; u \neq 0]$, *then*

$$|\langle 10, n, g\rangle, \boldsymbol{x}, 1| = \mathit{infimum}\{\mathit{supremum}[|g, y, \boldsymbol{x}, \alpha(y)|+1 : \mathit{all}\ y] \\ :(y)[\{g\}(y, \boldsymbol{x}) \to \alpha(y) \neq 0]\},$$

where α *varies over all one-place total functions on* B^* *to* B^*.

A moment's reflection will show that $|f, \boldsymbol{u}, z|_{\mathrm{h}}$ is the smallest ordinal at which we verify that $\{f\}_{\mathrm{h}}(\boldsymbol{u}) \to z$ in Case C10′ by this definition. Because of the supremum operation in this clause, the ordinals $|f, \boldsymbol{u}, z|_{\mathrm{h}}$ need not be finite, and may even be uncountable, if B is uncountable. Proofs by induction on $|f, \boldsymbol{u}, z|_{\mathrm{h}}$ will be transfinite inductions.

LEMMA 37. *If* $f \in PRI$, *then* $\{f\}_{\mathrm{pr}}(\boldsymbol{u}) = \{f\}_{\mathrm{h}}(\boldsymbol{u})$. *In particular* $PR(A, \boldsymbol{\varphi}) \subset HP(A, \boldsymbol{\varphi})$.

LEMMA 38. *If* $\mathrm{f} \in PR(A, \mathrm{g}_1, \ldots, \mathrm{g}_m, \boldsymbol{\varphi})$ *and for* $i = 1, \ldots, m$, $\mathrm{g}_i \in HP(A, \boldsymbol{\varphi})$, *then* $\mathrm{f} \in HP(A, \boldsymbol{\varphi})$.

LEMMA 39. *The functions* $S^m(f, y_1, \ldots, y_m)$ *of Lemma* 12 *satisfy*

$$\{f\}_{\mathrm{h}}(\boldsymbol{y}, \boldsymbol{x}) = \{S^m(f, \boldsymbol{y})\}_{\mathrm{h}}(\boldsymbol{x}). \tag{10.3}$$

Moreover, if $\{f\}_{\mathrm{h}}(\boldsymbol{y}, \boldsymbol{x}) \to z$, *then* $|f, \boldsymbol{y}, \boldsymbol{x}, z|_{\mathrm{h}} < |S^m(f, \boldsymbol{y}), \boldsymbol{x}, z|_{\mathrm{h}}$. (*The* S^m *theorem.*)

LEMMA 40. (a) *The function* $EV(f, k')$ *of Lemma* 20 *satisfies*

$$\{EV(f, k')\}_{\mathrm{h}}(\boldsymbol{u}, \boldsymbol{u}') = \{f\}_{\mathrm{h}}(\boldsymbol{u}). \tag{10.4}$$

Moreover, if $\{f\}_{\mathrm{h}}(\boldsymbol{u}) \to z$, *then* $|f, \boldsymbol{u}, z|_{\mathrm{h}} < |EV(f, k'), \boldsymbol{u}, \boldsymbol{u}', z|_{\mathrm{h}}$.

(b) *The function* $FV(f, k')$ *of Lemma* 20 *satisfies*

$$\{FV(f, k')\}_{\mathrm{h}}(\boldsymbol{u}', \boldsymbol{u}) = \{f\}_{\mathrm{h}}(\boldsymbol{u}). \tag{10.5}$$

Moreover, if $\{f\}_{\mathrm{h}}(\boldsymbol{u}) \to z$, *then* $|f, \boldsymbol{u}, z|_{\mathrm{h}} < |FV(f, k'), \boldsymbol{u}', \boldsymbol{u}, z|_{\mathrm{h}}$.

LEMMA 41. *If* $\mathrm{rc}(f)$ *is the function of Lemma* 21 *and* $m = \mathrm{rc}(f)$, *then*

$$\{f\}_{\mathrm{h}}(m, \boldsymbol{x}) = \{m\}_{\mathrm{h}}(\boldsymbol{x}). \tag{10.6}$$

(*The recursion theorem.*)

LEMMA 42. *There is a combinatorial function* $\mathrm{p}(f)$ *such that for each* f,

$$\{f\}_{\nu}(\boldsymbol{u}) = \{\mathrm{p}(f)\}_{\mathrm{h}}(\boldsymbol{u}). \tag{10.7}$$

In particular for each $A \subset B$, $SC(A, \boldsymbol{\varphi}) \subset HP(A, \boldsymbol{\varphi})$.

Proof is exactly like that of Lemma 31, where we add just one more case, C9, to the definition of $\mathrm{p}(f)$:

Case C9. $\{f\}(\boldsymbol{x}) = \nu y[\{g\}(y, \boldsymbol{x}) \to 0]$. Put $\mathrm{p}(f) = \langle 9, n, \mathrm{p}(g)\rangle$.

We then prove by induction on C0′–C9′ that if $\{f\}_\nu(\boldsymbol{u}) \to z$, then $\{\mathrm{p}(f)\}_{\mathrm{h}}(\boldsymbol{u}) \to z$, and by induction on $|f', \boldsymbol{u}, z|_{\mathrm{h}}$ that if $\{f'\}_{\mathrm{h}}(\boldsymbol{u}) \to z$ and $f' = \mathrm{p}(f)$, then $\{f\}_\nu(\boldsymbol{u}) \to z$. Consider the only new case: if $\{\mathrm{p}(f)\}_{\mathrm{h}}(\boldsymbol{x}) \to \boldsymbol{y}$ in Case C9, then $\mathrm{p}(f) = \langle 9, n, \mathrm{p}(g)\rangle$ and we must have $\{\mathrm{p}(g)\}_{\mathrm{h}}(y, \boldsymbol{x}) \to 0$; since $|\mathrm{p}(g), y, \boldsymbol{x}, 0|_{\mathrm{h}} < |\mathrm{p}(f), \boldsymbol{x}, y|_{\mathrm{h}}$ the ind. hyp. applies and $\{g\}_\nu(y, \boldsymbol{x}) \to 0$, i.e. $\{f\}_\nu(\boldsymbol{x}) \to y$.

Remark 14. The method of proof of Lemmas 31 and 42 clearly extends to prove the same result if we add further "reasonable" clauses to C0–C10, e.g. clauses introducing higher type objects other than $\mathbf{E}$; the same remark applies to the transitivity Lemmas 22, 32 and 43.

Lemma 43. *There is a combinatorial function* $\mathrm{tr}_{\mathrm{h}}(f, c)$ *such that if* $\chi(\boldsymbol{u}') = \{c\}_{\mathrm{h}}(\boldsymbol{u}')$ *and* $\mathrm{f}(\boldsymbol{u}) = \{f\}_{\mathrm{h}}(\chi, \boldsymbol{u})$, *then* $\mathrm{f}(\boldsymbol{u}) = \{\mathrm{tr}_{\mathrm{h}}(f, c)\}_{\mathrm{h}}(\boldsymbol{u})$. *Thus if* $\mathrm{f} \in HP(A, \mathrm{g}_1, \ldots, \mathrm{g}_m, \boldsymbol{\varphi})$ *and* $\mathrm{g}_1, \ldots, \mathrm{g}_m \in HP(A, \boldsymbol{\varphi})$, *then* $\mathrm{f} \in HP(A, \boldsymbol{\varphi})$. (*Transitivity lemma.*)

Proof is just like those of Lemmas 22 and 32. In the definition of $\mathrm{tr}_{\mathrm{h}}(f, c)$ we now add one more case to those already present in the definition of $\mathrm{tr}_\nu(f, c)$:

Case C10. $\{f\}(\chi, \boldsymbol{x}) = \mathbf{E}(\lambda y\{g\}(\chi, y, \boldsymbol{x}))$. Put $\mathrm{tr}_{\mathrm{h}}(f, c) = \langle 10, n, \mathrm{tr}_{\mathrm{h}}(g, c)\rangle$.

Proof that the function $\mathrm{tr}_{\mathrm{h}}(f, c)$ so defined satisfies the lemma is just like that of Lemma 22. Let us treat here just Case C10 of the implication analogous to (5.11): *if* $\{f'\}_{\mathrm{h}}(\boldsymbol{u}) \to z$ *and* $f' = \mathrm{tr}_{\mathrm{h}}(f, c)$, *then* $\{f\}_{\mathrm{h}}(\chi, \boldsymbol{u}) \to z$. This implication is proved by induction on $|f', \boldsymbol{u}, z|_{\mathrm{h}}$.

Case C10. Assume $\{f'\}(\boldsymbol{x}) = \mathbf{E}(\lambda y\{g'\}(y, \boldsymbol{x})) \to z$, where $f' = \mathrm{tr}_{\mathrm{h}}(f, c)$ and hence by the definition $g' = \mathrm{tr}_{\mathrm{h}}(g, c)$, where $f = \langle 10, n, g\rangle$. Let us suppose that $z = 1$ first. Then $(y)(Eu)[\{g'\}_{\mathrm{h}}(y, \boldsymbol{x}) \to u \ \& \ u \neq 0]$. By the definition of $|f', \boldsymbol{x}, z|_{\mathrm{h}}$, there is a one-place total function α on B^* to B^* such that $(y)[\{g'\}_{\mathrm{h}}(y, \boldsymbol{x}) \to \alpha(y) \neq 0]$ and

$$\textit{supremum}\ \{|g', y, \boldsymbol{x}, \alpha(y)|_{\mathrm{h}} + 1 : \textit{all}\ y\} \leqq |f', \boldsymbol{x}, z|;$$

in particular, for each y, $|g', y, \boldsymbol{x}, \alpha(y)|_{\mathrm{h}} < |f', \boldsymbol{x}, z|$, and the induction hypothesis applies to g', y, $\boldsymbol{x}$, $\alpha(y)$. Hence for each y, $\{g\}_{\mathrm{h}}(\chi, y, \boldsymbol{x}) \to \alpha(y) \neq 0$, so $\{f\}_{\mathrm{h}}(\chi, \boldsymbol{x}) = \mathbf{E}(\lambda y\{g\}_{\mathrm{h}}(\chi, y, \boldsymbol{x})) \to 1$. The argument is similar when $z = 0$.

Lemma 44. *If the predicates* $P(\boldsymbol{u})$, $Q(\boldsymbol{u})$, $R(y, \boldsymbol{u})$ *and the single-valued totally defined function* $\mathrm{f}(\boldsymbol{u})$ *are in* $HP(A, \boldsymbol{\varphi})$, *then so are the predicates* $P(\boldsymbol{u})\ \&\ Q(\boldsymbol{u})$, $\bar{P}(\boldsymbol{u})$, $(Ey)R(y, \boldsymbol{u})$ *and* $R(\mathrm{f}(\boldsymbol{u}), \boldsymbol{u})$. *Thus* $\bigcup_n \sigma_n^0(A, \boldsymbol{\varphi}) \subset HP(A, \boldsymbol{\varphi})$, *i.e. every predicate projective from A is hyperprojective from A.*

Remark 15. Instead of introducing the object $\mathbf{E} = \mathbf{E}_{B^*}$, one may introduce an object $\mathbf{E}_C$ for each $C \subset B^*$ that embodies quantification over C; interesting examples are $C = \omega$ or $C = B$. Such computabilities relative to restricted quantification are best studied in the context of abstract higher order computability which we are not considering here.

In studying search computability we went to great lengths to avoid assuming that the predicate $x=y$ was necessarily computable. Our motivation was partly that this would be an unnatural assumption and partly that in several interesting examples the assumption would be false. In all the natural examples that we know $x=y$ is hyperprojective, and in fact falls quite low in the projective hierarchy. On the other hand, one easily constructs artificial examples where $x=y$ is not hyperprojective. In any case, it seems to us that there is no a priori reason why $x=y$ must be hyperprojective, so at the cost of a little extra work we shall develop the hyperprojective hierarchy without this assumption. To the best of our knowledge we lose no interesting theorems in this way. On the other hand we get as a by-product a classification of $x=y$ in the projective hierarchy, whenever $x=y$ is hyperprojective.

Put

$$u \sim v \Leftrightarrow (f)[\{f\}_{\mathrm{pr}}(u) \cap \omega = \{f\}_{\mathrm{pr}}(v) \cap \omega]. \tag{10.8}$$

For sequences $\boldsymbol{u}$, $\boldsymbol{u}'$ we put

$$u_1, \ldots, u_k \sim u_1', \ldots, u_k' \Leftrightarrow u_1 \sim u_1' \,\&\cdots\&\, u_k \sim u_k' \tag{10.9}$$

and for subsets C, E of B^*,

$$C \sim E \Leftrightarrow (u)(Ev)[u \in C \Rightarrow v \in E \,\&\, u \in E \Rightarrow v \in C \,\&\, u \sim v]. \tag{10.10}$$

We recall that if $\boldsymbol{\varphi}$ consists of single-valued total functions then each $\{f\}_{\mathrm{pr}}(\boldsymbol{u})$ is single-valued and total. It is then easy to see that

(10.11) *If $\boldsymbol{\varphi}$ is single-valued and total, then $u \sim v \Leftrightarrow$ for every primitive computable predicate $P(x)$, $P(u) \Leftrightarrow P(v)$.*

The next lemma implies in particular that if no primitive computable predicate can distinguish between u and v, then no hyperprojective predicate can distinguish between u and v.

LEMMA 45. *Assume that $\boldsymbol{\varphi}$ consists of single-valued, totally defined functions. If $f, \boldsymbol{u} \sim f', \boldsymbol{u}'$ (i.e. $f \sim f'$, $u \sim u_1', \ldots, u_k \sim u_k'$) and $\{f\}_{\mathrm{h}}(\boldsymbol{u}) \to z$, then there exists a $z' \sim z$ such that $\{f'\}_{\mathrm{h}}(\boldsymbol{u}') \to z'$.*

Proof is by induction on C0′–C10′ after a few preliminary results.

(10.12) *If $u \sim u'$, then $\pi u \sim \pi u'$, $\delta u \sim \delta u'$, $(v, u) \sim (v, u')$ and $(u, v) \sim (u', v)$, for all v.*

To prove the last of the assertions as an example, we must show that for each f, $\{f\}_{\mathrm{pr}}((u, v)) \cap \omega = \{f\}_{\mathrm{pr}}((u', v)) \cap \omega$. Choose e so that $\{e\}_{\mathrm{pr}}(v, u) = \{f\}_{\mathrm{pr}}((u, v))$ and notice that $\{e\}_{\mathrm{pr}}(v, u) = \{S^1(e, v)\}_{\mathrm{pr}}(u)$; since $u \sim u'$ then we must have

$$\{S^1(e, v)\}_{\mathrm{pr}}(u) \cap \omega = \{S^1(e, v)\}_{\mathrm{pr}}(u') \cap \omega,$$

which proves the result.

(10.13) *If* $u_1 \sim u_1', \ldots, u_k \sim u_k'$, *then* $\langle u_1, \ldots, u_k\rangle \sim \langle u_1', \ldots, u_k'\rangle$.

We first show by induction on n and the definition (1.9) of $s_n(x_0, \ldots, x_n)$, that if $x_0 \sim x_0', \ldots, x_n \sim x_n'$, then $s_n(x_0, \ldots, x_n) \sim s_n(x_0', \ldots, x_n')$ and then use the definition (1.14) of $\langle u_1, \ldots, u_n\rangle$.

(10.14) $$\textit{If } u \sim u', \textit{ then } (u)_i \sim (u')_i.$$

Easy by induction on i using (10.12) and the definition (1.10).

(10.15) $$\textit{If } u \sim u' \textit{ and } \mathrm{Seq}\,(u), \textit{ then } \mathrm{Seq}\,(u'), (u)_0 = (u')_0 \textit{ and } (i)_{i \leq (u)_0}[(u)_i \sim (u')_i].$$

This is immediate since Seq (u) is primitive computable and $(u)_0 \in \omega$ when Seq (u).

We give some of the cases in the inductive proof of the lemma. Because of (10.15), from the assumption $f, \boldsymbol{u} \sim f', \boldsymbol{u}'$ we know that $\{f\}(\boldsymbol{u})$ and $\{f'\}(\boldsymbol{u}')$ are defined by the same case hypothesis. Cases C1–C4 are immediate from the preliminary results.

Case C0. $\boldsymbol{u} = t_1, \ldots, t_{n_i}, \boldsymbol{x}$; $\boldsymbol{u}' = t_1', \ldots, t_{n_i}', \boldsymbol{x}'$; $\{f\}_{\mathrm{h}}(\boldsymbol{u}) = \varphi_i(t_1, \ldots, t_{n_i}) = z$. We must show that $\{f'\}_{\mathrm{h}}(\boldsymbol{u}') = \varphi_i(t_1', \ldots, t_{n_i}') = z' \sim z$. Assume $z' \not\sim z$; there exists then some e, so that $\{e\}_{\mathrm{pr}}(z) \cap \omega \neq \{e\}_{\mathrm{pr}}(z') \cap \omega$. Choose m so that

$$\{m\}_{\mathrm{pr}}(t) = \{e\}_{\mathrm{pr}}(\varphi_i((t)_1, \ldots, (t)_{n_i}));$$

we have then

$$\{m\}_{\mathrm{pr}}(\langle t_1, \ldots, t_{n_i}\rangle) = \{e\}_{\mathrm{pr}}(z); \{m\}_{\mathrm{pr}}(\langle t_1', \ldots, t_{n_i}'\rangle) = \{e\}_{\mathrm{pr}}(z'),$$

so that $\{m\}_{\mathrm{pr}}(\langle t_1, \ldots, t_{n_i}\rangle) \cap \omega \neq \{m\}_{\mathrm{pr}}(\langle t_1', \ldots, t_{n_i}'\rangle) \cap \omega$, so that $\langle t_1, \ldots, t_{n_i}\rangle \not\sim \langle t_1', \ldots, t_{n_i}'\rangle$, violating (10.13).

Case C5. $\boldsymbol{u} = \boldsymbol{x}$; $\boldsymbol{u}' = \boldsymbol{x}'$; $\{f\}_{\mathrm{h}}(\boldsymbol{x}) = \{g\}_{\mathrm{h}}(\{h\}_{\mathrm{h}}(\boldsymbol{x}), \boldsymbol{x})$.
Now there is a u, such that $\{h\}_{\mathrm{h}}(\boldsymbol{x}) \to u$ and $\{g\}_{\mathrm{h}}(u, \boldsymbol{x}) \to z$. By ind. hyp. there exist $u' \sim u$ and $z' \sim z$ such that $\{h'\}_{\mathrm{h}}(\boldsymbol{x}') \to u'$ and $\{g'\}_{\mathrm{h}}(u', \boldsymbol{x}') \to z'$. (Here we are using (10.15), according to which $h' = (f')_4 \sim (f)_4 = h$, $g' = (f')_3 \sim (f)_3 = g$.)

Case C6 is treated just like Case C5 and Cases C7 and C8 are trivial.

Case C9. $\boldsymbol{u} = \boldsymbol{x}$; $\boldsymbol{u}' = \boldsymbol{x}'$; $\{f\}_{\mathrm{h}}(\boldsymbol{x}) = \nu y[\{g\}_{\mathrm{h}}(y, \boldsymbol{x}) \to 0]$. If $\{f\}_{\mathrm{h}}(\boldsymbol{x}) \to z$, then $\{g\}_{\mathrm{h}}(z, \boldsymbol{x}) \to 0$. By ind. hyp. there is then a t such that $t \sim 0$ and $\{g'\}_{\mathrm{h}}(z, \boldsymbol{x}') \to t$; but if $t \sim 0$ then $t = 0$ since the predicate $x = 0$ is primitive computable, hence $\{g'\}_{\mathrm{h}}(z, \boldsymbol{x}') \to 0$ and we can take $z' = z$.

Case C10. $\boldsymbol{u} = \boldsymbol{x}$; $\boldsymbol{u}' = \boldsymbol{x}'$; $\{f\}_{\mathrm{h}}(\boldsymbol{x}) = \mathbf{E}(\lambda y\{g\}_{\mathrm{h}}(y, \boldsymbol{x}))$.
Assume $\{f\}_{\mathrm{h}}(\boldsymbol{x}) \to 1$. Then for each y there is a $u \neq 0$ such that $\{g\}_{\mathrm{h}}(y, \boldsymbol{x}) \to u$. Hence by ind. hyp. for each y there is a $u' \sim u \neq 0$, so that $\{g'\}_{\mathrm{h}}(y, \boldsymbol{x}) \to u'$. Since $u' \sim u \neq 0$ clearly implies $u' \neq 0$, the result is immediate. The argument for the case $\{f\}_{\mathrm{h}}(\boldsymbol{x}) \to 0$ is similar.

LEMMA 46. *Assume that* φ *consists of total single-valued functions.* (a) *The predicate* $u \sim v$ *is in* π_1^0, *i.e. there exists an absolutely primitive computable* $R(y, u, v)$ *such that*

$$u \sim v \Leftrightarrow (y)R(y, u, v).$$

(b) *If the predicate* $u = v$ *is hyperprojective, then*

$$u = v \Leftrightarrow u \sim v,$$

so that $u = v$ *is in* π_1^0.

Proof. Using Lemmas 1, 4, 11 and 25 and Theorem 1, we have

$$\begin{aligned} u \sim v \Leftrightarrow (f)(y)(y')[f \in PRI_1 \ \&\ T(f, u, y) \ \&\ T(f, v, y') \\ \Rightarrow [[U(f, u, y) \in \omega \Leftrightarrow U(f, v, y') \in \omega] \\ \&\ [U(f, u, y) \in \omega \Rightarrow U(f, u, y) = U(f, v, y')]]]. \end{aligned}$$

To prove (b) assume that $u = v$ is hyperprojective, say with characteristic function $\chi_=(u, v)$. By Lemma 45 then, if $u \sim v$ we have $u, u \sim u, v$ so $\chi_=(u, u) \sim \chi_=(u, v)$, i.e. $\chi_=(u, v) \sim 0$, i.e. $\chi_=(u, v) = 0$, i.e. $u = v$.

REMARK 16. The lemma shows that the predicate $u = v$ must be either search computable, or in π_1^0, or not occur in the hyperprojective hierarchy at all.

EXAMPLE 6. Suppose that $B \neq \varnothing$, take φ to be empty, $b \in B$. Using the notation in the proof of Lemma 5, we have shown that if $\{f\}_{\text{pr}}(u_1, \ldots, u_k) \in \omega$, then $\{f\}_{\text{pr}}(u_1, \ldots, u_k) = \{f^\#\}_{\text{pr}}(u_1^\#, \ldots, u_k^\#)$. This implies that if $u^\# = v^\#$, then for every f such that $\{f\}_{\text{pr}}(u) \in \omega$, we have $\{f\}_{\text{pr}}(u) = \{f^\#\}_{\text{pr}}(u^\#) = \{f^\#\}_{\text{pr}}(v^\#) = \{f\}_{\text{pr}}(v)$, i.e. $u \sim v$. Conversely if $u \sim v$ and we put $\mathrm{f}(x) = \mathrm{c}(x^\#)$ in the notation of (3.10), we must have $\mathrm{f}(u) = \mathrm{f}(v)$, since for each x, $\mathrm{f}(x) \in \omega$, so that $u^\# = v^\#$. Thus in this example $u \sim v \Leftrightarrow u^\# = v^\#$ and if B has more than one element, then the predicate $u \sim v$ is distinct from the predicate $u = v$. Hence $u = v$ is not hyperprojective.

11. **The universal set H.** We call a predicate $P(\boldsymbol{u})$ *semi-hyperprojective from* A, if there is a p.m.v. function $\mathrm{f}(\boldsymbol{u})$ in $HP(A, \varphi)$ such that

$$P(\boldsymbol{u}) \Leftrightarrow \mathrm{f}(\boldsymbol{u})\downarrow; \tag{11.1}$$

this is clearly equivalent to the condition

$$P(\boldsymbol{u}) \Leftrightarrow \mathrm{g}(\boldsymbol{u}) \to 0, \tag{11.2}$$

if we take $\mathrm{g}(\boldsymbol{u}) = 0 \cdot \mathrm{f}(\boldsymbol{u})$. A set is semi-hyperprojective if its representing predicate is. These sets and predicates are the abstract analogs of the Π_1^1 sets and predicates in ordinary recursion theory. In this section we define a set H, which we show to be *absolutely semi-hyperprojective* (i.e. semi-hyperprojective from $\varnothing$) and complete in the class of semi-hyperprojective predicates. Later on we shall attach ordinals to the members of H and see that it is the abstract analog of Kleene's O in [5] or Spector's W in [21].

From this section on we make the blanket assumption that $\boldsymbol{\varphi}$ *consists of total, single-valued functions.* Put

$$\boldsymbol{\varphi}^{\sim} = \chi_{\sim}, \chi_1, \ldots, \chi_l, \tag{11.3}$$

where $\chi_{\sim}$ is the characteristic function of $u \sim v$ and χ_i is the characteristic function of $\varphi_i(t_1, \ldots, t_{n_i}) \sim z$,

$$\chi_i(t_1, \ldots, t_{n_i}, z) = 0 \Leftrightarrow \varphi_i(t_1, \ldots, t_{n_i}) \sim z. \tag{11.4}$$

We shall utilize extensively functions which are primitive computable in the list $\boldsymbol{\varphi}^{\sim}$ rather than the list $\boldsymbol{\varphi}$, so for simplicity we adopt the two notations:

$$\{f\}_{\mathrm{pr}}^{\sim}(\boldsymbol{\varphi}, \boldsymbol{u}) = \{f\}_{\mathrm{pr}}^{\sim}(\boldsymbol{u}) = \{f\}_{\mathrm{pr}}(\boldsymbol{\varphi}^{\sim}, \boldsymbol{u}), \tag{11.5}$$

$$e_v = \{e\}_{\mathrm{pr}}^{\sim}(v) \qquad (e \in PRI_1). \tag{11.6}$$

The set H is defined inductively by the following three clauses:

(11.7)

(a) $\langle 1, 0\rangle \in H$,

(b) *if* $e \in PRI_1$ & $(v)[e_v \in H]$, *then* $\langle 2, e\rangle \in H$,

(c) *if* $e \in PRI_1$ & $(Ev)[e_v \in H]$, *then* $\langle 3, e\rangle \in H$,

(d) $x \in H$ *only by* (a)–(c).

Lemma 47. *There exist combinatorial functions* $\mathrm{P}_{\&}(x, y)$ *and* $P_{\vee}(x, y)$ *such that*

$$x \in H \;\&\; y \in H \Leftrightarrow \mathrm{P}_{\&}(x, y) \in H, \tag{11.8}$$

$$x \in H \vee y \in H \Leftrightarrow \mathrm{P}_{\vee}(x, y) \in H. \tag{11.9}$$

Proof. Choose $m \in 0^*$ so that

$$\begin{aligned} \{m\}_{\mathrm{pr}}^{\sim}(x, y, v) &= x \quad \text{if } v = 0, \\ &= y \quad \text{if } v \neq 0, \end{aligned}$$

and put $\mathrm{P}_{\&}(x, y) = \langle 2, S^2(m, x, y)\rangle$. Similarly for $\mathrm{P}_{\vee}(x, y)$.

Theorem 6. *There is a function* $\mathrm{q}(f, u, z)$, *absolutely primitive computable in the list* $\boldsymbol{\varphi}^{\sim}$ (*defined by* (11.3)), *such that*:

$$\text{if } \{f\}_{\mathrm{h}}(u_1, \ldots, u_k) \to z, \text{ then } \mathrm{q}(f, \langle u_1, \ldots, u_k\rangle, z) \in H, \tag{11.10}$$

$$\text{if } \mathrm{q}(f, \langle u_1, \ldots, u_k\rangle, z) \in H, \text{ then there exists a } z' \sim z \text{ such that } \{f\}_{\mathrm{h}}(u_1, \ldots, u_k) \to z'. \tag{11.11}$$

In particular, $\mathrm{q}(f, u, z)$ *is absolutely hyperprojective in* $\boldsymbol{\varphi}$ *and*

$$z \in \omega \Rightarrow [\{f\}_{\mathrm{h}}(u_1, \ldots, u_k) \to z \Leftrightarrow \mathrm{q}(f, \langle u_1, \ldots, u_k\rangle, z) \in H]. \tag{11.12}$$

Proof. We shall abbreviate $\langle u_1, \ldots, u_k\rangle$ by $\langle \boldsymbol{u}\rangle$. The function $\mathrm{q}(f, u, z)$ will be defined by the recursion theorem for primitive computable functions, Lemma 14, from an index q of itself.

The definition of $\mathrm{q}(f, u, z)$ from q, f, u, z will be by cases C0–C10 (and an Otherwise case), according to the definition of $\{f\}_{\mathrm{h}}(u_1, \ldots, u_k)$ that the form of f suggests (where $k=(f)_2$, $\mathrm{Seq}\,(u)$, $(u)_0=k$ and for $1 \leqq j \leqq k$, $u_j=(u)_j$). We shall identify the case hypotheses in the manner of the proof of Lemma 22 and several later lemmas. Many of the necessary index constructions will be omitted since they can be effected routinely with the techniques of §4.

Simultaneously with the definition we give in each case the argument for the proof of (11.10) by induction on C0′–C10′; we shall prove (11.11) at the end.

Case C0. $\boldsymbol{u}=t_1, \ldots, t_{n_i}, \boldsymbol{x}$; $\{f\}(\boldsymbol{u})=\varphi_i(t_1, \ldots, t_{n_i})$. Put

$$\begin{aligned} \mathrm{q}(f, u, z) &= \langle 1, 0\rangle && \text{if } z \sim \varphi_i(t_1, \ldots, t_{n_i}), \\ &= 0 && \text{otherwise.} \end{aligned}$$

(The case hypothesis here is the condition on f stated in Case C0 of Lemma 11 together with

$$\mathrm{Seq}\,(u)\ \&\ (u)_0 = (f)_2; \tag{11.13}$$

the notation "$\boldsymbol{u}=t_1, \ldots, t_{n_i}, \boldsymbol{x}$" above simply reminds us that in the definition to follow the sequence $(u)_1, \ldots, (u)_{(f)_2}$ will be denoted by "$t_1, \ldots, t_{n_i}, x_1, \ldots, x_n$". There are l subcases to this case, depending on $i=(f)_3$. In the ith subcase the definition above written without any notational conventions would be

$$\begin{aligned} \mathrm{q}(f, u, z) &= \langle 1, 0\rangle && \text{if } z \sim \varphi_i((u)_1, \ldots, (u)_{n_i}), \\ &= 0 && \text{otherwise.} \end{aligned}$$

We shall identify the other case hypotheses in the same manner, hoping that this makes the motivation for the treatment of each case clear. It should be a routine matter to produce each time the combinatorial conditions on f, u and z that are implied and then to state the definition directly in terms of q, f, u, z, without the use of extra names for these objects.)

It is immediate in this case, that if $\{f\}(\boldsymbol{u}) \to z$, then $\mathrm{q}(f, \langle u_1, \ldots, u_k\rangle, z)=\langle 1, 0\rangle \in H$.

The other direct cases C1–C4 are handled in exactly the same way and we omit them.

Case C5. $\boldsymbol{u}=\boldsymbol{x}$, $\{f\}(\boldsymbol{x})=\{g\}(\{h\}(\boldsymbol{x}), \boldsymbol{x})$.

Using the methods of §4 we can choose a combinatorial function $\mathrm{m}_0(q)$ such that

$$\{\mathrm{m}_0(q)\}^{\sim}_{\mathrm{pr}}(f, u, z, v) = \mathrm{P}_{\&}(\{q\}^{\sim}_{\mathrm{pr}}(h, u, v), \{q\}^{\sim}_{\mathrm{pr}}(g, \langle v, (u)_1, \ldots, (u)_{(u)_0}\rangle, z))$$

(where $g=(f)_3$, $h=(f)_4$) and then put $\mathrm{q}(f, u, z)=\langle 3, S^3(\mathrm{m}_0(q), f, u, z)\rangle$. Assume that $\{f\}_{\mathrm{h}}(\boldsymbol{x}) \to z$. Then for some v, $\{h\}_{\mathrm{h}}(\boldsymbol{x}) \to v$ and $\{g\}_{\mathrm{h}}(v, \boldsymbol{x}) \to z$, so by induction hypothesis $\mathrm{q}(h, \langle \boldsymbol{x}\rangle, v) \in H$ and $\mathrm{q}(g, \langle v, \boldsymbol{x}\rangle, z) \in H$, hence by Lemma 47, $\mathrm{q}(f, \langle \boldsymbol{x}\rangle, z) \in H$.

Case C6. *Subcase* (a). $\boldsymbol{u}=y, \boldsymbol{x}$, $\{f\}(y, \boldsymbol{x})=\{g\}(y, \boldsymbol{x})$, $y \in B^0$.

Choose a combinatorial function $\mathrm{m}_1(q)$ such that $\{\mathrm{m}_1(q)\}^{\sim}_{\mathrm{pr}}(f, u, z, v)= \{q\}^{\sim}_{\mathrm{pr}}(g, u, z)$ and put $\mathrm{q}(f, u, z)=\langle 2, S^3(\mathrm{m}_1(q), f, u, z)\rangle$. Assume that $\{f\}_{\mathrm{h}}(y, \boldsymbol{x})$

$\to z$; then $\{g\}_{\mathrm{h}}(y, \boldsymbol{x}) \to z$, so by ind. hyp. $\mathrm{q}(g, \langle y, \boldsymbol{x}\rangle, z) \in H$, hence for each v, $\{S^3(\mathrm{m}_1(q), f, \langle y, \boldsymbol{x}\rangle, z)\}^{\sim}_{\mathrm{pr}}(v) \in H$, hence $\mathrm{q}(f, \langle y, \boldsymbol{x}\rangle, z) \in H$. (For proving (11.10) the simpler definition $\mathrm{q}(f, u, z) = \mathrm{q}(g, u, z)$ would be sufficient in this case. We need this slightly more complicated definition of $\mathrm{q}(f, u, z)$ in order to get a simple proof of (11.11) in this case.)

Case C6. *Subcase* (b). $\boldsymbol{u} = (s, t), \boldsymbol{x}, \{f\}((s, t), \boldsymbol{x}) = \{h\}(\{f\}(s, \boldsymbol{x}), \{f\}(t, \boldsymbol{x}), s, t, \boldsymbol{x})$. This case is handled very similarly to Case C5 and we omit the details.

Case C7. $\boldsymbol{u} = \boldsymbol{x}, \{f\}(\boldsymbol{x}) = \{g\}(x_{j+1}, x_1, \ldots, x_j, x_{j+2}, \ldots, x_n)$.
Choose a combinatorial $\mathrm{m}_2(q, f)$ such that

$$\{\mathrm{m}_2(q,f)\}^{\sim}_{\mathrm{pr}}(f, u, z, v) = \{q\}^{\sim}_{\mathrm{pr}}(g, \langle (u)_{j+1}, (u)_1, \ldots, (u)_j, (u)_{j+2}, \ldots, (u)_n\rangle, z)$$

and then proceed as in Case C6, *Subcase* (a) to put $\mathrm{q}(f, u, z) = \langle 2, S^3(\mathrm{m}_2(q, f), f, u, z)\rangle$.

Case C8. $\boldsymbol{u} = e, \boldsymbol{x}, \boldsymbol{y}, \{f\}(e, \boldsymbol{x}, \boldsymbol{y}) = \{e\}(\boldsymbol{x})$.
Choose a combinatorial $\mathrm{m}_3(q, f)$ such that

$$\{\mathrm{m}_3(q,f)\}^{\sim}_{\mathrm{pr}}(f, u, z, v) = \{q\}^{\sim}_{\mathrm{pr}}((u)_1, \langle (u)_2, \ldots, (u)_{n+1}\rangle, z)$$

and put $\mathrm{q}(f, u, z) = \langle 2, S^3(\mathrm{m}_3(q, f), f, u, z)\rangle$ as before.

Assume $\{f\}_{\mathrm{h}}(e, \boldsymbol{x}, \boldsymbol{y}) \to z$; then $\{e\}_{\mathrm{h}}(\boldsymbol{x}) \to z$, so by ind. hyp. $\mathrm{q}(e, \langle \boldsymbol{x}\rangle, z) \in H$, hence for each v, $\{S^3(\mathrm{m}_3(q, f), f, \langle e, \boldsymbol{x}, \boldsymbol{y}\rangle, z)\}^{\sim}_{\mathrm{pr}}(v) = \mathrm{q}(e, \langle \boldsymbol{x}\rangle, z) \in H$, hence $\mathrm{q}(f, \langle e, \boldsymbol{x}, \boldsymbol{y}\rangle, z) \in H$.

Case C9. $\boldsymbol{u} = \boldsymbol{x}, \{f\}(\boldsymbol{x}) = \nu y[\{g\}(y, \boldsymbol{x}) \to 0]$.
Choose a combinatorial $\mathrm{m}_4(q)$ such that

$$\{\mathrm{m}_4(q)\}^{\sim}_{\mathrm{pr}}(f, u, z, v) = \{q\}^{\sim}_{\mathrm{pr}}(g, \langle z, (u)_1, \ldots, (u)_n\rangle, 0)$$

and then proceed as before.

Case C10. $\boldsymbol{u} = \boldsymbol{x}, \{f\}(\boldsymbol{x}) = \mathbf{E}(\lambda y\{g\}(y, \boldsymbol{x}))$.
Choose a combinatorial $\mathrm{m}_5(q)$ such that

$$\{\mathrm{m}_5(q)\}^{\sim}_{\mathrm{pr}}(f, u, v, v') = \{q\}^{\sim}_{\mathrm{pr}}(g, \langle v, (u)_1, \ldots, (u)_n\rangle, v'),$$

then choose a combinatorial $\mathrm{m}_6(q)$ such that

$$\{\mathrm{m}_6(q)\}^{\sim}_{\mathrm{pr}}(f, u, v) = \langle 3, S^3(\mathrm{m}_5(q), f, u, v)\rangle$$

and put $\mathrm{m}_7(q, f, u) = \langle 2, S^2(\mathrm{m}_6(q), f, u)\rangle$. It is now easy to verify that if $\{f\}(\boldsymbol{x})$ is defined, then $\mathrm{m}_7(q, f, \langle \boldsymbol{x}\rangle) \in H$.

Then choose $\mathrm{m}_8(q)$ so that

$$\{\mathrm{m}_8(q)\}^{\sim}_{\mathrm{pr}}(f, u, v) = \{q\}^{\sim}_{\mathrm{pr}}(g, \langle v, (u)_1, \ldots, (u)_n\rangle, 0),$$

$\mathrm{m}_9(q)$ so that

$$\begin{aligned}\{\mathrm{m}_9(q)\}^{\sim}_{\mathrm{pr}}(f, u, v, v') &= \{q\}^{\sim}_{\mathrm{pr}}(g, \langle v, (u)_1, \ldots, (u)_n\rangle, v') \quad \text{if } v' \neq 0,\\ &= 0 \quad \text{if } v' = 0,\end{aligned}$$

and $\mathrm{m}_{10}(q)$ so that $\{\mathrm{m}_{10}(q)\}_{\mathrm{pr}}^{\sim}(f, u, v) = \langle 3, S^3(\mathrm{m}_9(q), f, u, v)\rangle$, and put

$$\begin{aligned} \mathrm{q}(f, u, z) &= \mathrm{P}_{\&}(\mathrm{m}_7(q, f, u), \langle 3, S^2(\mathrm{m}_8(q), f, u)\rangle) \quad \text{if } z = 0, \\ &= \mathrm{P}_{\&}(\mathrm{m}_7(q, f, u), \langle 2, S^2(\mathrm{m}_{10}(q), f, u)\rangle) \quad \text{if } z = 1, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Assume that $\{f\}_{\mathrm{h}}(\boldsymbol{x}) \to 1$. Then for each v there is a $v' \neq 0$ so that $\{g\}_{\mathrm{h}}(v, \boldsymbol{x}) \to v'$, so by ind. hyp. $\mathrm{q}(g, \langle v, \boldsymbol{x}\rangle, v') \in H$, hence $\{\mathrm{m}_9(q)\}_{\mathrm{pr}}^{\sim}(f, \langle \boldsymbol{x}\rangle, v, v') \in H$. Thus for each v, $\{\mathrm{m}_{10}(q)\}_{\mathrm{pr}}^{\sim}(f, \langle \boldsymbol{x}\rangle, v) \in H$, hence $\langle 2, S^2(\mathrm{m}_{10}(q), f, \langle \boldsymbol{x}\rangle)\rangle \in H$, hence $\mathrm{q}(f, \langle \boldsymbol{x}\rangle, 1) \in H$. A similar argument holds when $\{f\}(\boldsymbol{x}) \to 0$.

Otherwise. Put $\mathrm{q}(f, u, z) = 0$.

It is clear that $\mathrm{q}(f, u, z)$ is absolutely primitive computable in $\boldsymbol{\varphi}^{\sim}$ and we have sketched the proof of (11.10). We prove (11.11) by induction on $\mathrm{q}(f, \langle u_1, \ldots, u_k\rangle, z) \in H$. First we notice that if $\mathrm{q}(f, \langle u_1, \ldots, u_k\rangle, z) \in H$, then one of Cases C0–C10 must apply to the definition of $\mathrm{q}(f, \langle u_1, \ldots, u_k\rangle, z)$, since $0 \notin H$.

Case C0. We must have $\mathrm{q}(f, \langle t_1, \ldots, t_n, \boldsymbol{x}\rangle, z) = \langle 1, 0\rangle$, since $0 \notin H$, hence $z \sim \varphi_i(t_1, \ldots, t_{n_i})$ and we can take $z' = \varphi_i(t_1, \ldots, t_{n_i})$.

Cases C1–C4 are handled similarly.

Case C5. If $\mathrm{q}(f, \langle \boldsymbol{x}\rangle, z) \in H$, we must have that for some v, $\{\mathrm{m}_0(q)\}_{\mathrm{pr}}^{\sim}(f, \langle \boldsymbol{x}\rangle, z, v) \in H$, hence by (11.8) we must have that for some v $\mathrm{q}(h, \langle \boldsymbol{x}\rangle, v) \in H$ and $\mathrm{q}(g, \langle v, \boldsymbol{x}\rangle, z) \in H$. Hence by ind. hyp. there exist $v' \sim v$ and $z' \sim z$ such that $\{h\}_{\mathrm{h}}(\boldsymbol{x}) \to v'$ and $\{g\}_{\mathrm{h}}(v, \boldsymbol{x}) \to z'$. By Lemma 45 now there must exist a $z'' \sim z'$ such that $\{g\}_{\mathrm{h}}(v', \boldsymbol{x}) \to z''$, which is the required result.

Similar uses of Lemma 45 are needed in treating some of the other cases. We outline only one more such argument.

Case C10. Assume $\mathrm{q}(f, \langle \boldsymbol{x}\rangle, z) \in H$. We have then that $\mathrm{m}_7(q, f, \langle \boldsymbol{x}\rangle) \in H$ by Lemma 47, hence for each v, $\langle 3, S^3(\mathrm{m}_5(q), f, \langle \boldsymbol{x}\rangle, v)\rangle \in H$, hence for each v there exists a v' so that $\mathrm{q}(g, \langle v, \boldsymbol{x}\rangle, v') \in H$, hence by ind. hyp. for each v there exists a $v'' \sim v'$ such that $\{g\}_{\mathrm{h}}(v, \boldsymbol{x}) \to v''$, hence $\{f\}_{\mathrm{h}}(\boldsymbol{x})$ is defined. To complete the proof we now take cases on $z = 0$ or $z = 1$ and see that in each case $\{f\}_{\mathrm{h}}(\boldsymbol{x})$ takes on the correct value.

12. **Ordinals and initial segments.** We assign ordinals to the elements of H by the three inductive clauses below, following the general procedure described in §5 for assigning ordinals to the objects satisfying an inductive definition.

$$\begin{aligned} &\text{(a)} \quad |\langle 1, 0\rangle| = 0, \\ (12.1) \qquad &\text{(b)} \quad |\langle 2, e\rangle| = \textit{supremum}\,\{|e_v| + 1 : v \in B^*\}, \\ &\text{(c)} \quad |\langle 3, e\rangle| = \textit{infimum}\,\{|e_v| + 1 : v \in B^*,\ e_v \in H\}. \end{aligned}$$

Put

$$(12.2) \qquad \kappa = \textit{supremum}\,\{|x| : x \in H\},$$

and set

$$\text{(12.3)} \qquad \textit{if } x \notin H, \textit{ then } |x| = \kappa.$$

In this section we show that the initial segments of H, i.e. sets of the form $\{x : |x| \leqq |y|\}$ are hyperprojective uniformly for $y \in H$. Our approach through a minimum function is that of [17] (where again it is inspired from Gandy's [3]) except that now we have to deal with the multiple-valued computations of hyperprojective functions. Theorem 7 below is the main tool for establishing the basic properties of the hyperprojective hierarchy in §13–§15.

One could avoid introducing the universal set H and instead prove the analog of Theorem 7 for the ordinals $|f, \boldsymbol{u}, z|_{\mathrm{h}}$ that we associate with finite sequences f, $\boldsymbol{u}$, z when $\{f\}_{\mathrm{h}}(\boldsymbol{u}) \to z$. However the three simple clauses of the inductive definition of H are much easier to deal with than clauses C0–C10, both here and in the representation theorems of §§17–19.

We recall our blanket assumption that $\boldsymbol{\varphi}$ *consists of single-valued, total functions.*

THEOREM 7. *There is an absolutely hyperprojective function* $\mathrm{p}(x, y)$ *such that*:

$$\text{(12.4)} \qquad \textit{if } |x| < \kappa \textit{ and } |x| \leqq |y|, \textit{ then } \mathrm{p}(x, y) = 0,$$

$$\text{(12.5)} \qquad \textit{if } |y| < |x|, \textit{ then } \mathrm{p}(x, y) = 1.$$

In particular,

$$\text{(12.6)} \qquad x \in H \vee y \in H \Rightarrow \mathrm{p}(x, y) \textit{ is defined and single-valued},$$

$$\text{(12.7)} \qquad y \in H \Rightarrow [|x| \leqq |y| \Leftrightarrow \mathrm{p}(x, y) = 0].$$

Proof. The function $\mathrm{p}(x, y)$ will be defined from an index p by the recursion theorem for hyperprojective functions, Lemma 41. There are sixteen cases to the definition, labelled ij $(i, j = 1, 2, 3, 4)$. Our labelling code is that in case ij x appears to be in H by virtue of the ith clause of definition (11.7), if $i = 1, 2, 3$, and x is obviously not in H if $i = 4$. (Similarly j codes the case for y.) The only nontrivial cases are 22, 23, 32 and 33.

Cases $1j$, $j = 1, 2, 3, 4$. $x = \langle 1, 0\rangle$. Put $\mathrm{p}(x, y) = 0$.

Cases $i1$, $i = 2, 3, 4$. $y = \langle 1, 0\rangle$ & $x \neq \langle 1, 0\rangle$. Put $\mathrm{p}(x, y) = 1$.

Cases $i4$, $i = 2, 3, 4$. $y \neq \langle 1, 0\rangle$ & $\neg[\mathrm{Seq}\,(y)$ & $(y)_0 = 2$ & $2 \leqq (y)_1 \leqq 3$ & $(y)_2 \in PRI_1]$. Put $\mathrm{p}(x, y) = 0$.

Cases $4j$, $j = 2, 3$. $\mathrm{Seq}\,(y)$ & $(y)_0 = 2$ & $2 \leqq (y)_1 \leqq 3$ & $(y)_2 \in PRI_1$ & $x \neq \langle 1, 0\rangle$ & $\neg[\mathrm{Seq}\,(x)$ & $(x)_0 = 2$ & $2 \leqq (x)_1 \leqq 3$ & $(x)_2 \in PRI_1]$.
Put $\mathrm{p}(x, y) = 1$.

For each of the remaining cases we assume the following condition, which we do not repeat each time:

$$\mathrm{Seq}\,(x) \;\&\; (x)_0 = 2 \;\&\; (x)_2 \in PRI_1 \;\&\; 2 \leqq (x)_1 \leqq 3 \;\&$$
$$\mathrm{Seq}\,(y) \;\&\; (y)_0 = 2 \;\&\; (y)_2 \in PRI_1 \;\&\; 2 \leqq (y)_1 \leqq 3.$$

We put $e = (x)_2$, $m = (y)_2$.

Case 22. $(x)_1=2$ & $(y)_1=2$. Put

$$\begin{aligned} \mathrm{p}(x, y) &= 0 \quad \text{if } \bar{\mathrm{s}}\bar{\mathrm{g}}(\mathbf{E}(\lambda v \bar{\mathrm{s}}\bar{\mathrm{g}}(\mathbf{E}(\lambda v'\{p\}_{\mathrm{h}}(e_v, m_{v'}))))) \to 0, \\ &= 1 \quad \text{if } \mathbf{E}(\lambda v \bar{\mathrm{s}}\bar{\mathrm{g}}(\mathbf{E}(\lambda v'\{p\}_{\mathrm{h}}(e_v, m_{v'})))) \to 0. \end{aligned}$$

Case 23. $(x)_1=2$ & $(y)_1=3$. Put

$$\begin{aligned} \mathrm{p}(x, y) &= 0 \quad \text{if } \bar{\mathrm{s}}\bar{\mathrm{g}}(\mathbf{E}(\lambda v \bar{\mathrm{s}}\bar{\mathrm{g}}(\{p\}_{\mathrm{h}}(e_{(v)_0}, m_{(v)_1})))) \to 0, \\ &= 1 \quad \text{if } (Ev')[\mathbf{E}(\lambda v \bar{\mathrm{s}}\bar{\mathrm{g}}(\{p\}_{\mathrm{h}}(e_v, m_{v'}))) \to 0]. \end{aligned}$$

(Here we use the notation $(Ev')[\mathrm{f}(v') \to 0]$ for $\nu v'[\mathrm{f}(v') \to 0]$ *is defined*, i.e. $0 \cdot \nu v'[\mathrm{f}(v') \to 0] \to 0$.)

Case 32. $(x)_1=3$ & $(y)_1=2$. Put

$$\begin{aligned} \mathrm{p}(x, y) &= 0 \quad \text{if } (Ev)[\mathbf{E}(\lambda v'\{p\}_{\mathrm{h}}(e_v, m_{v'})) \to 0], \\ &= 1 \quad \text{if } \bar{\mathrm{s}}\bar{\mathrm{g}}(\mathbf{E}(\lambda v\{p\}_{\mathrm{h}}(e_{(v)_0}, m_{(v)_1}))) \to 0. \end{aligned}$$

Case 33. $(x)_1=3$ & $(y)_1=3$. Put

$$\begin{aligned} \mathrm{p}(x, y) &= 0 \quad \text{if } (Ev)[\bar{\mathrm{s}}\bar{\mathrm{g}}(\mathbf{E}(\lambda v' \bar{\mathrm{s}}\bar{\mathrm{g}}(\{p\}_{\mathrm{h}}(e_v, m_{v'})))) \to 0], \\ &= 1 \quad \text{if } (Ev')[\bar{\mathrm{s}}\bar{\mathrm{g}}(\mathbf{E}(\lambda v\{p\}_{\mathrm{h}}(e_v, m_{v'}))) \to 0]. \end{aligned}$$

We prove (12.4) and (12.5) by transfinite induction on *infimum* $(|x|, |y|) < \kappa$. Since the result is immediate when $x=\langle 1, 0\rangle$ or $y=\langle 1, 0\rangle$, we only need argue when $\mathrm{p}(x, y)$ is defined by Cases 22, 23, 32 and 33.

Case 22. Assume first that $x \in H$, $|x| \leqq |y|$. Since then for all v, $e_v \in H$ and $|e_v| < |x|$, the induction hypothesis guarantees that $\lambda v v'\{p\}_{\mathrm{h}}(e_v, m_{v'})$ is completely defined and single-valued, so that $\mathrm{p}(x, y)$ is defined and single-valued. Since

$$\textit{supremum}\ \{|e_v|+1\} \leqq \textit{supremum}\ \{|m_{v'}|+1\},$$

$(v)(Ev')[|e_v| \leqq |m_{v'}|]$, so by induction hypothesis $(v)(Ev')[\{p\}_{\mathrm{h}}(e_v, m_{v'})=0]$, from which one easily verifies that $\mathrm{p}(x, y) \to 0$.

Now assume that $|y| < |x|$; then *supremum* $\{|m_{v'}|+1\} <$ *supremum* $\{|e_v|+1\}$, hence $(Ev)(v')[|m_{v'}| < |e_v|]$. Thus by induction hypothesis $\lambda v v'\{p\}_{\mathrm{h}}(e_v, m_{v'})$ is single-valued, so that $\mathrm{p}(x, y)$ is single-valued, and $(Ev)(v')[\{p\}_{\mathrm{h}}(e_v, m_{v'})=1]$, which easily leads to the conclusion $\mathrm{p}(x, y) \to 1$.

Case 23. Assume first that $x \in H$, $|x| \leqq |y|$, so that *supremum* $\{|e_v|+1\} \leqq$ *infimum* $\{|m_{v'}|+1\}$. Then by induction hypothesis, for all v, $\{p\}_{\mathrm{h}}(e_{(v)_0}, m_{(v)_1})=0$, which easily leads to the conclusion $\mathrm{p}(x, y) \to 0$. Towards a contradiction assume that also $\mathrm{p}(x, y) \to 1$; then

$$(Ev')[\mathbf{E}(\lambda v \bar{\mathrm{s}}\bar{\mathrm{g}}(\{p\}_{\mathrm{h}}(e_v, m_{v'}))) \to 0], \text{ i.e. } (Ev')(Ev)[\{p\}_{\mathrm{h}}(e_v, m_{v'}) \to 1],$$

which by induction hypothesis means $(Ev')(Ev)[|m_{v'}| < |e_v|]$, violating the condition $|x| \leqq |y|$ above.

Now assume that $|y| < |x|$, i.e. $(Ev')(Ev)[m_{v'}| < |e_v|]$. Using the induction hypothesis this implies easily that $\mathrm{p}(x, y) \to 1$. It is impossible to also have $\mathrm{p}(x, y)$

$\to 0$, because then for all v, $\{p\}_{\mathrm{h}}(e_{(v)_0}, m_{(v)_1}) \to 0$, which violates the induction hypothesis if we pick $(v)_0$ and $(v)_1$ so that $|m_{(v)_1}| < |e_{(v)_0}|$.
Case 32 is symmetric to Case 23 and is handled similarly.

Case 33. Assume first that $x \in H$ and $|x| \leqq |y|$, so that

$$\textit{infimum}\,\{|e_v|+1\} \leqq \textit{infimum}\,\{|m_{v'}|+1\}.$$

Then $(Ev)[e_v \in H \;\&\; (v')[|e_v| \leqq |m_{v'}|]]$, so by induction hypothesis

$$(Ev)(v')[\{p\}_{\mathrm{h}}(e_v, m_{v'}) = 0];$$

this easily implies that $\mathrm{p}(x, y) \to 0$. Towards a contradiction assume that also $\mathrm{p}(x, y) \to 1$. Then $(Ev')(v)[\{p\}_{\mathrm{h}}(e_v, m_{v'}) \to 1]$; if we pick v such that $|e_v| < |x|$ $\&\; (v')[|e_v| \leqq |m_{v'}|]$, then by induction hypothesis $(v')[\{p\}_{\mathrm{h}}(e_v, m_{v'})=0]$, which yields a contradiction.

Proof in this case that if $|y| < |x|$, then $\mathrm{p}(x, y)=1$ is symmetric and will be omitted.

13. **Semi-hyperprojective predicates.** For a fixed $A \subset B^*$, let $sHP(A)=sHP(A, \boldsymbol{\varphi})$ be the class of all predicates semi-hyperprojective from A:

(13.1) *$P(\boldsymbol{u})$ is in $sHP(A)$ if for some $f \in A^*$,*
$$P(\boldsymbol{u}) \Leftrightarrow \{f\}_{\mathrm{h}}(\boldsymbol{u}) \to 0.$$

Put

(13.2) $$H^A = H \cap A^*;\; \kappa^A = \textit{supremum}\,\{|x| : x \in H^A\}.$$

In this notation, clearly, $\kappa^B=\kappa$. In general we would expect $\kappa^A<\kappa$.

In many of the results below we shall need the assumption "A is hyperprojective from A". We are mostly interested in the cases $A=\varnothing$, $A=B$ and A finite, when this condition is of course satisfied.

LEMMA 48. (a) *Suppose $P(\boldsymbol{u})$, $Q(\boldsymbol{u})$ and $R(y, \boldsymbol{u})$ are in $sHP(A)$, let $\mathrm{f}(\boldsymbol{u})$ be total and single-valued in $HP(A)$. Then the predicates $P(\boldsymbol{u})$ & $Q(\boldsymbol{u})$, $P(\boldsymbol{u}) \vee Q(\boldsymbol{u})$, $(Ey)R(y, \boldsymbol{u})$, $(y)R(y, \boldsymbol{u})$ and $R(f(\boldsymbol{u}), \boldsymbol{u})$ are in $sHP(A)$.*

(b) *$P(\boldsymbol{u})$ is in $HP(A)$ if and only if both $P(\boldsymbol{u})$ and $\bar{P}(\boldsymbol{u})$ are in $sHP(A)$.*

Proof. To prove two of the assertions of (a), if χ_R is the characteristic function of R, then

$$(Ey)R(y, \boldsymbol{u}) \Leftrightarrow 0\cdot \nu y[\chi_R(y, \boldsymbol{u}) \to 0] \to 0,$$
$$(y)R(y, \boldsymbol{u}) \Leftrightarrow \bar{\mathrm{s}}\bar{\mathrm{g}}(\mathbf{E}(\lambda y \bar{\mathrm{s}}\bar{\mathrm{g}}(\chi_R(y, \boldsymbol{u})))) \to 0.$$

To prove (b) in the nontrivial direction, assume that χ_1 and χ_2 are the functions in $HP(A)$ determining $P(\boldsymbol{u})$ and $\bar{P}(\boldsymbol{u})$ respectively as in (13.1). Then by multiple-valued cases,

$$\begin{aligned} \chi_P(\boldsymbol{u}) &= 0 \quad \text{if } \chi_1(\boldsymbol{u}) \to 0,\\ &= 1 \quad \text{if } \chi_2(\boldsymbol{u}) \to 0. \end{aligned}$$

We recall our blanket assumption that $\boldsymbol{\varphi}$ *consists of single-valued total functions.*

LEMMA 49. *Assume A is in $HP(A)$. Then the set H^A is not in $HP(A)$.*

Proof. We obtain a contradiction from the assumption that H^A is hyperprojective from A. By Theorem 6,

$$\{f\}_{\mathrm{h}}(f) \to 0 \Leftrightarrow \mathrm{q}(f, \langle f \rangle, 0) \in H; \tag{13.3}$$

since $\mathrm{q}(f, u, z)$ is absolutely primitive computable in the list $\boldsymbol{\varphi}^{\sim}$ of (11'.3) which consists of integer-valued functions, Lemma 1 applies and we know that

$$f \in A^* \Rightarrow \mathrm{q}(f, \langle f \rangle, 0) \in A^*. \tag{13.4}$$

Hence,

$$f \in A^* \Rightarrow [\{f\}_{\mathrm{h}}(f) \to 0 \Leftrightarrow \mathrm{q}(f, \langle f \rangle, 0) \in H^A]. \tag{13.5}$$

Since H^A is hyperprojective from A, the function

$$\begin{aligned} \mathrm{g}(x) &= 1 \quad \text{if } \mathrm{q}(x, \langle x \rangle, 0) \in H^A \\ &= 0 \quad \text{if } \mathrm{q}(x, \langle x \rangle, 0) \notin H^A, \end{aligned}$$

is hyperprojective from A, say with index $g \in A^*$. But then we have

$$\{g\}_{\mathrm{h}}(g) = 1 \Rightarrow \mathrm{q}(g, \langle g \rangle, 0) \in H^A \Rightarrow \{g\}_{\mathrm{h}}(g) \to 0$$

and

$$\{g\}_{\mathrm{h}}(g) = 0 \Rightarrow \mathrm{q}(g, \langle g \rangle, 0) \notin H^A \Rightarrow \neg(\{g\}_{\mathrm{h}}(g) \to 0),$$

which is a contradiction.

THEOREM 8. (a) *There is a combinatorial function* $\mathrm{eh}(m, \boldsymbol{u})$ *such that a predicate $P(\boldsymbol{u})$ is in $sHP(A)$ only if there is an $m \in A^*$ such that*

$$P(u) \Leftrightarrow \mathrm{eh}(m, \boldsymbol{u}) \in H. \tag{13.6}$$

(b) *Assume A is in $HP(A)$, let $P(\boldsymbol{u})$ be in $sHP(A)$, let* $\mathrm{f}(\boldsymbol{u})$ *be any single-valued total function in $HP(A)$ such that*

$$P(u) \Leftrightarrow \mathrm{f}(\boldsymbol{u}) \in H. \tag{13.7}$$

Then $P(\boldsymbol{u})$ is in $HP(A)$ if and only if

$$\textit{supremum}\, \{|\mathrm{f}(\boldsymbol{u})| : P(\boldsymbol{u})\} < \kappa^A. \tag{13.8}$$

(*Boundedness theorem.*)

Proof. To prove (a), choose $g \in 0^*$ so that

$$\{g\}^{\sim}_{\mathrm{pr}}(m, \boldsymbol{u}, v) = \mathrm{q}(m, \langle \boldsymbol{u} \rangle, 0) \tag{13.9}$$

and put

$$\mathrm{eh}(m, \boldsymbol{u}) = \langle 2, S^{k+1}(g, m, \boldsymbol{u}) \rangle. \tag{13.10}$$

Now given $P(\boldsymbol{u})$ in $sHP(A)$, choose $m \in A^*$ such that $\mathrm{P}(\boldsymbol{u}) \Leftrightarrow \{m\}_{\mathrm{h}}(\boldsymbol{u}) \rightarrow 0$; then

$$P(\boldsymbol{u}) \Leftrightarrow \mathrm{q}(m, \langle \boldsymbol{u} \rangle, 0) \in H \Leftrightarrow (v)[\{S^{k+1}(g, m, \boldsymbol{u})\}_{\mathrm{pr}}^{\sim}(v) \in H] \Leftrightarrow \mathrm{eh}(m, \boldsymbol{u}) \in H.$$

To prove (b) one way, suppose (13.8) holds, choose $z \in H^A$ so that

$$\mathit{supremum}\,\{|f(\boldsymbol{u})| : P(\boldsymbol{u})\} \leqq |z|.$$

Then $P(\boldsymbol{u}) \Leftrightarrow \mathrm{p}(\mathrm{f}(\boldsymbol{u}), z) = 0$ where $\mathrm{p}(x, y)$ is the function of Theorem 7, so $P(\boldsymbol{u})$ is in $HP(A)$.

For the other direction suppose that $\mathit{supremum}\,\{|\mathrm{f}(\boldsymbol{u})| : P(\boldsymbol{u})\} \geqq \kappa^A$ but that nevertheless $P(\boldsymbol{u})$ is in $HP(A)$. Then

$$x \in H^A \Leftrightarrow x \in A^* \;\&\; (E\boldsymbol{u})[P(\boldsymbol{u}) \;\&\; \mathrm{p}(x, \mathrm{f}(\boldsymbol{u})) = 0]$$

which implies easily that H^A is in $HP(A)$, contradicting Lemma 49.

LEMMA 50. (a) *H is absolutely semi-hyperprojective.* (b) *If A is in $HP(A)$, then H^A is in $sHP(A)$.*

Proof. The predicate $\{f\}_{\mathrm{h}}(f) \rightarrow 0$ is in sHP; it is easy to see that it is not hyperprojective, for if it were we could define

$$\begin{aligned} \mathrm{g}(f) &= 1 \quad \text{if } \{f\}_{\mathrm{h}}(f) \rightarrow 0, \\ &= 0 \quad \text{if } \neg(\{f\}_{\mathrm{h}}(f) \rightarrow 0), \end{aligned}$$

and choosing g so that $\mathrm{g}(f) = \{g\}_{\mathrm{h}}(f)$ we could easily get a contradiction by considering $\mathrm{g}(g)$. Thus $\{f\}_{\mathrm{h}}(f) \rightarrow 0 \Leftrightarrow \mathrm{h}(f) \in H$ for some absolutely primitive computable $\mathrm{h}(f)$, and

$$\mathit{supremum}\,\{|\mathrm{h}(f)| : \{f\}_{\mathrm{h}}(f) \rightarrow 0\} = \kappa$$

by Theorem 8. Hence

$$x \in H \Leftrightarrow (Ef)[\{f\}_{\mathrm{h}}(f) \rightarrow 0 \;\&\; \mathrm{p}(x, \mathrm{h}(f)) \rightarrow 0], \tag{13.11}$$

where $\mathrm{p}(x, y)$ is the function of Theorem 7, which easily shows that H is in sHP.

(b) is immediate.

LEMMA 51. *Let $P(\boldsymbol{u})$ and $Q(\boldsymbol{u})$ be in $sHP(A)$. There exist predicates $P_1(\boldsymbol{u})$, $Q_1(\boldsymbol{u})$ in $sHP(A)$ such that*

$$\begin{aligned} &P_1(\boldsymbol{u}) \Rightarrow P(\boldsymbol{u}),\; Q_1(\boldsymbol{u}) \Rightarrow Q(\boldsymbol{u}), \\ &P(\boldsymbol{u}) \vee Q(\boldsymbol{u}) \Rightarrow P_1(\boldsymbol{u}) \vee Q_1(\boldsymbol{u}), \\ &\neg(P_1(\boldsymbol{u}) \;\&\; Q_1(\boldsymbol{u})). \end{aligned} \tag{13.12}$$

(*Reduction lemma.*)

Proof. Put

$$\begin{aligned} P_1(\boldsymbol{u}) &\Leftrightarrow P(\boldsymbol{u}) \;\&\; \mathrm{p}(\mathrm{f}(\boldsymbol{u}), \mathrm{g}(\boldsymbol{u})) \rightarrow 0, \\ Q_1(\boldsymbol{u}) &\Leftrightarrow Q(\boldsymbol{u}) \;\&\; \mathrm{p}(\mathrm{f}(\boldsymbol{u}), \mathrm{g}(\boldsymbol{u})) \rightarrow 1, \end{aligned}$$

where $\mathrm{f}(\boldsymbol{u})$ and $\mathrm{g}(\boldsymbol{u})$ are chosen in $PR(A)$ by Theorem 8 so that

$$P(u) \Leftrightarrow \mathrm{f}(\boldsymbol{u}) \in H, \qquad Q(\boldsymbol{u}) \Leftrightarrow \mathrm{g}(\boldsymbol{u}) \in H,$$

and $\mathrm{p}(x, y)$ is the function of Theorem 7.

LEMMA 52. *Let* $\mathrm{f}(\boldsymbol{u})$ *be a totally defined multiple-valued function in* $HP(A)$ *all of whose values are integers. Then* $\mathrm{f}(\boldsymbol{u})$ *has a single-valued branch in* $HP(A)$, *i.e. there is a total single-valued function* $\mathrm{g}(\boldsymbol{u})$ *in* $HP(A)$ *such that* $\mathrm{g} \subset \mathrm{f}$. (*Integer-valued single-valuedness lemma.*)

Proof. Put

$$\begin{aligned} P(\boldsymbol{u}, i) \Leftrightarrow \mathrm{f}(\boldsymbol{u}) \to i \ \&\ (j)[\mathrm{p}(\mathrm{q}(f, \langle \boldsymbol{u} \rangle, i), \mathrm{q}(f, \langle \boldsymbol{u} \rangle, j)) \to 0] \\ \&\ (j)[j < i \Rightarrow \mathrm{p}(\mathrm{q}(f, \langle \boldsymbol{u} \rangle, j), \mathrm{q}(f, \langle \boldsymbol{u} \rangle, i)) \to 1], \\ \mathrm{g}(\boldsymbol{u}) = \nu i P(\boldsymbol{u}, i), \end{aligned}$$

where $\mathrm{f}(\boldsymbol{u}) = \{f\}_{\mathrm{h}}(\boldsymbol{u})$.

LEMMA 53. *Let* $\mathrm{f}(\boldsymbol{u})$ *be a totally defined multiple-valued function in* $HP(A)$. *There exists a totally defined multiple-valued function* $\mathrm{g}(\boldsymbol{u})$ *in* $HP(A)$ *such that*

$$(13.13) \qquad \begin{aligned} &\mathrm{g}(\boldsymbol{u}) \to z \Rightarrow (Ez')[z' \sim z \ \&\ \mathrm{f}(\boldsymbol{u}) \to z'], \\ &\textit{the predicate } \mathrm{g}(\boldsymbol{u}) \to z \textit{ is in } HP(A). \end{aligned}$$

Similarly with functionals, if $\mathrm{f}(\mathrm{g}_1, \ldots, \mathrm{g}_m, \boldsymbol{u})$ *is totally defined in* $HP(A)$, *there is a totally defined* $\mathrm{g}(\mathrm{g}_1, \ldots, \mathrm{g}_m, \boldsymbol{u})$ *in* $HP(A)$ *such that*

$$(13.14) \qquad \begin{aligned} &\mathrm{g}(\mathrm{g}_1, \ldots, \mathrm{g}_m, \boldsymbol{u}) \to \mathrm{z} \Rightarrow (Ez')[z' \sim \mathrm{z} \ \&\ \mathrm{f}(\mathrm{g}_1, \ldots, \mathrm{g}_m, \boldsymbol{u}) \to z'], \\ &\textit{the predicate } \mathrm{g}(\mathrm{g}_1, \ldots, \mathrm{g}_m, \boldsymbol{u}) \to z \textit{ is in } HP(A). \end{aligned}$$

(*The normal branch lemma.*)

Proof. Let $\mathrm{f}(\boldsymbol{u}) = \{f\}_{\mathrm{h}}(\boldsymbol{u})$, choose $m \in 0^*$ such that

$$\{m\}_{\mathrm{pr}}^{\sim}(f, \boldsymbol{u}, z) = \mathrm{q}(f, \langle \boldsymbol{u} \rangle, z).$$

Since $\mathrm{f}(\boldsymbol{u})$ is totally defined, for each $\boldsymbol{u}$ there is a z such that $\mathrm{f}(\boldsymbol{u}) \to z$, so for each $\boldsymbol{u}$ there is a z such that $\{m\}_{\mathrm{pr}}^{\sim}(f, \boldsymbol{u}, z) \in H$. Hence, for each $\boldsymbol{u}$,

$$(13.15) \qquad \langle 3, S^{k+1}(m, f, \boldsymbol{u}) \rangle \in H$$

and

$$(13.16) \qquad (Ez)[\mathrm{f}(\boldsymbol{u}) \to z \ \&\ |\mathrm{q}(f, \langle \boldsymbol{u} \rangle, z)| \leqq |\langle 3, S^{k+1}(m, f, \boldsymbol{u})|].$$

Thus we put

$$\mathrm{g}(\boldsymbol{u}) = \nu z[\mathrm{p}(\mathrm{q}(f, \langle u \rangle, z), \langle 3, S^{k+1}(m, f, \boldsymbol{u}) \rangle) \to 0]$$

and the argument above shows that $\mathrm{g}(\boldsymbol{u})$ is total. Now if $\mathrm{g}(\boldsymbol{u}) \to z$, then $\mathrm{q}(\mathrm{f}, \langle \boldsymbol{u} \rangle, z) \in H$, so by Theorem 6 there is a $z' \sim z$ so that $\mathrm{f}(\boldsymbol{u}) \to z'$.

The lemma for functionals follows by exactly the same argument relative to the list $\boldsymbol{\varphi}' = \mathrm{g}_1, \ldots, \mathrm{g}_m, \boldsymbol{\varphi}$, for each fixed m-tuple $\mathrm{g}_1, \ldots, \mathrm{g}_m$ of functions.

REMARK 17. There seems to be no hope of assigning to each total, hyperprojective $\mathrm{f}(\boldsymbol{u})$ a total, single-valued hyperprojective $\mathrm{g}(\boldsymbol{u})$ such that $\mathrm{g} \subset \mathrm{f}$ without using some explicit well-ordering on B^*. Lemma 53 seems to go as far as possible in this direction and will in fact be very useful in the next section.

If $\mathrm{f}(\boldsymbol{u})$ is total, single-valued and hyperprojective and if the predicate $x = y$ is hyperprojective, then the predicate $\mathrm{f}(\boldsymbol{u}) = z$ is hyperprojective: because $\mathrm{f}(\boldsymbol{u}) = z \Leftrightarrow \mathrm{q}(f, \langle \boldsymbol{u} \rangle, z) \in H$ and $\mathrm{f}(\boldsymbol{u}) \neq z \Leftrightarrow (Ez')[z' \neq z \;\&\; \mathrm{f}(\boldsymbol{u}) = z']$. On the other hand, if $x = y$ is not hyperprojective and $\mathrm{f}(x) = x$, then $\mathrm{f}(x) = z$ is not hyperprojective.

It is immediate from Lemma 46 that if $x = y$ is hyperprojective and $\mathrm{g}(\boldsymbol{u})$ is associated with $\mathrm{f}(\boldsymbol{u})$ as in Lemma 53, then $\mathrm{g} \subset \mathrm{f}$.

REMARK 18. Assume that $\boldsymbol{\varphi}$ is empty. We show that in this case a set of integers is semi-hyperprojective if and only if it is in Π^1_1, i.e. reducible to the set O of [5]. It will follow that if $\boldsymbol{\varphi}$ is empty, then the functions on ω to ω which are hyperprojective are exactly the hyperarithmetic functions.

Let $\{e\}_{\mathrm{K}}(x)$ be the eth partial recursive function in the sense of [4] $(e, x \in \omega)$, put

$$(13.17) \qquad \begin{aligned} \chi(e, x) &= \{e\}_{\mathrm{K}}(x) && \textit{if } e, x \in \omega,\ \{e\}_{\mathrm{K}}(x) \textit{ is defined}, \\ &= 0 && \textit{if } e, x \in \omega,\ \{e\}_{\mathrm{K}}(x) \textit{ not defined}, \\ &= 1 && \textit{if } e \notin \omega \textit{ or } x \notin \omega. \end{aligned}$$

We define a set S of natural numbers by the inductive clauses:

$$(13.18) \qquad \begin{aligned} &\text{(a)}\ \ 1 \in S, \\ &\text{(b)}\ \ \textit{if for all } x \in \omega,\ \{e\}_{\mathrm{K}}(x) \in S, \textit{ then } 3^e \in S. \end{aligned}$$

It is a straightforward exercise in the theory of Π^1_1 sets to show that S is a complete Π^1_1 set, e.g. by reducing O to it.

For the fixed list $\boldsymbol{\varphi} = \chi$ consisting of one function only, choose a combinatorial $\mathrm{m}(p)$ such that

$$(13.19) \qquad \{\mathrm{m}(p)\}^{\sim}_{\mathrm{pr}}(e, v) = \{p\}^{\sim}_{\mathrm{pr}}(\chi(e, v))$$

and define the function $\mathrm{p}(x)$ by the recursion theorem for primitive computable functions Lemma 14 from an index p as follows:

$$(13.20) \qquad \begin{aligned} \mathrm{p}(x) &= \langle 1, 0 \rangle && \text{if } x = 1, \\ &= \langle 2, S^1(\mathrm{m}(p), e) \rangle && \text{if } x = 3^e,\ e \in \omega, \\ &= 0 && \text{otherwise}. \end{aligned}$$

We prove by induction on (13.18) that

$$(13.21) \qquad x \in S \Rightarrow \mathrm{p}(x) \in H,$$

where H is defined by (11.7) for the list $\boldsymbol{\varphi}=\chi$. Clearly $\mathrm{p}(1)=\langle 1, 0\rangle \in H$, and if $3^e \in S$, then for all $v \in \omega$, $\mathrm{p}(\{e\}_{\mathrm{K}}(v))=\{p\}^{\sim}_{\mathrm{pr}}(\chi(e, v))=\{S^1(\mathrm{m}(p), e)\}^{\sim}_{\mathrm{pr}}(v) \in H$, and since for $v \notin \omega$ we have $\{S^1(\mathrm{m}(p), e)\}^{\sim}_{\mathrm{pr}}(v)=\langle 1, 0\rangle \in H$, it follows that $\mathrm{p}(3^e)= \langle 2, S^1(\mathrm{m}(p), e)\rangle \in H$. Conversely, we can easily show by induction on the definition of H that

$$\mathrm{p}(x) \in H \Rightarrow x \in S. \tag{13.22}$$

Thus by Lemma 50 the set S is absolutely semi-hyperprojective in χ, i.e.

$$x \in S \Leftrightarrow \mathrm{f}(\chi, x) \to 0$$

for some absolutely hyperprojective $\mathrm{f}(\chi, x)$. Now since χ is absolutely hyperprojective in the empty $\boldsymbol{\varphi}$, S is absolutely semi-hyperprojective in the empty $\boldsymbol{\varphi}$ by Lemma 43.

Now suppose that $B\neq \varnothing$, let $b \in B$ be fixed. We define a predicate $\{f\}^{\#}_{\mathrm{h}}(\boldsymbol{u}) \to z$ on finite sequences f, $\boldsymbol{u}$, z of b^* by restricting all variables in clauses C0′–C10′ to elements of b^*. In particular C10′ becomes

C10′#. *If for all $y \in b^*$ there is a $u \in b^*$ such that $\{g\}^{\#}(y, \boldsymbol{x}) \to u$ and if there is a $y \in b^*$ such that $\{g\}^{\#}(y, \boldsymbol{x}) \to 0$, then $\{\langle 10, n, g\rangle\}^{\#}(\boldsymbol{x}) \to 0$. If for all $y \in b^*$ there is a $u \in b^*$, $u\neq 0$, such that $\{g\}^{\#}(y, \boldsymbol{x}) \to u$, then $\{\langle 10, u, g\rangle\}^{\#}(\boldsymbol{x}) \to 1$.*

It is now easy to show by induction on C0′–C10′ that

$$\textit{if } \{f\}_{\mathrm{h}}(\boldsymbol{u}) \to z, \textit{ then } \{f^{\#}\}^{\#}_{\mathrm{h}}(\boldsymbol{u}^{\#}) \to z^{\#}, \tag{13.23}$$

where $x^{\#}$ is defined by (3.12) in the proof of Lemma 5, and $\boldsymbol{u}^{\#}=u_1^{\#}, \ldots, u_{\mathrm{k}}^{\#}$. Conversely we can show by induction on C0′#–C10′# that

$$\begin{aligned} &\textit{if } \{f^{\#}\}^{\#}_{\mathrm{h}}(\boldsymbol{u}^{\#}) \to z^{\#}, \textit{ then there exists a } z' \textit{ such that} \\ &\qquad\qquad z^{\#} = z'^{\#} \textit{ and } \{f\}_{\mathrm{h}}(\boldsymbol{u}) \to z'. \end{aligned} \tag{13.24}$$

We gave cases C5′# and C10′# of this induction as examples.

Case C5′#. Suppose $\{h^{\#}\}^{\#}_{\mathrm{h}}(\boldsymbol{x}) \to u^{\#}$ and $\{g^{\#}\}^{\#}_{\mathrm{h}}(u^{\#}, \boldsymbol{x}^{\#}) \to z^{\#}$. By ind. hyp. there exist u' and z' such that $u^{\#}=u'^{\#}$ and $z^{\#}=z'^{\#}$ and $\{h\}_{\mathrm{h}}(\boldsymbol{x}) \to u'$, $\{g\}_{\mathrm{h}}(u, \boldsymbol{x}) \to z'$. Now from Example 6 we know that if $\boldsymbol{\varphi}$ is empty then

$$u \sim v \Leftrightarrow u^{\#} = v^{\#}.$$

Hence by Lemma 45 there is a $z''\sim z'$, i.e. $z''^{\#}=z'^{\#}$ so that $\{g\}_{\mathrm{h}}(u', \boldsymbol{x}) \to z''$ from which it follows that $\{f\}_{\mathrm{h}}(\boldsymbol{x}) \to z''\sim z$.

Case C10′#. Suppose that for every $y \in b^*$ there is a $u \in b^*$, $u\neq 0$ such that $\{g^{\#}\}^{\#}_{\mathrm{h}}(y, \boldsymbol{x}^{\#}) \to u$. Then for each $y \in B^*$ there is a $u\neq 0$ such that $\{g^{\#}\}^{\#}_{\mathrm{h}}(y^{\#}, \boldsymbol{x}^{\#}) \to u$, so by ind. hyp. for each $y \in B^*$ there is a $u'\sim u$ (hence as above, $u'\neq 0$) such that $\{g\}_{\mathrm{h}}(y, \boldsymbol{x}) \to u'$. Thus $\{f\}_{\mathrm{h}}(\boldsymbol{x}) \to 1$. The argument is similar if $\{f\}_{\mathrm{h}}(\boldsymbol{x}) \to 0$ in this case.

Now from (13.23) and (13.24) it follows that if $\{f\}_{\mathrm{h}}(\boldsymbol{u})$ is single-valued on b^* and maps b^* into b^*, then

$$u_1, \ldots, u_{\mathrm{k}} \in b^* \Rightarrow \{f\}_{\mathrm{h}}(u_1, \ldots, u_{\mathrm{k}}) = \{f^{\#}\}^{\#}_{\mathrm{h}}(u_1, \ldots, u_{\mathrm{k}}).$$

To the predicate $P(f, \mathbf{u}, z) \Leftrightarrow \{f\}^{\#}_{\mathrm{n}}(\mathbf{u}) \to z$ on b^* we can associate a number-theoretic predicate $cP(f', \mathbf{u}', z')$ by the method of the proof of Lemma 5, which then by an analysis of inductive definitions like that of Wang in [24] is seen to be Π^1_1. It follows that every predicate of natural numbers which is semi-hyperprojective in the empty $\boldsymbol{\varphi}$ is Π^1_1, and hence that every hyperprojective function on ω to ω is hyperarithmetic.

Proof is a little simpler if $B = \varnothing$.

14. **Reduced ordinals.** For each $A \subset B^*$, each $x \in H^A$, put

$$|x|^A = \textit{order type of } \{|y| : y \in H^A \;\&\; |y| < |x|\}, \tag{14.1}$$

and put

$$\lambda^A = \textit{supremum}\, \{|x|^A : x \in H^A\}. \tag{14.2}$$

Clearly $|x|^B = |x|$ for each x and $\lambda^B = \kappa$. We call the ordinals $|x|^A$ the *reduced ordinals* of H^A. In general we expect $|x|^A$ to be much smaller than $|x|$; e.g. if A is denumerable then H^A is denumerable, so each $|x|^A$ is denumerable, while in general $|x|$ is uncountable.

In this section we characterize λ^A (when A is hyperprojective from A) as the smallest ordinal not realizable by a partial well-ordering which is hyperprojective from A.

We recall our blanket assumption that $\boldsymbol{\varphi}$ *consists of total, single-valued functions.*

LEMMA 54. *Assume that A is hyperprojective from A. If $x \in H^A$ and $x' \sim x$, then $x' \in H^A$ and $|x'| = |x|$.*

Proof. Since A is hyperprojective from A, it is in particular hyperprojective, hence A^* is hyperprojective. Hence by Lemma 45, if $x \in A^*$ and $x \sim x'$, then $x' \in A^*$. Thus it is enough to show that $x \in H \;\&\; x' \sim x \Rightarrow x' \in H \;\&\; |x'| = |x|$.

Now

$$x \in H \Leftrightarrow \mathrm{f}(x) \to 0, \qquad |u| \leqq |v| \Leftrightarrow \mathrm{p}(u, v) \to 0 \quad (u, v \in H),$$

for suitable absolutely hyperprojective $\mathrm{f}(x)$, $\mathrm{p}(u, v)$, so again by Lemma 45, if $\mathrm{f}(x) \to 0$ and $x \sim x'$ then $\mathrm{f}(x') \to 0$, so $x' \in H$. Also, the relation of x' to any $u \in H$ is the same as that of x, since by Lemma 45 $\mathrm{p}(u, x) \to 0 \Leftrightarrow \mathrm{p}(u, x') \to 0$, hence $|x'| = |x|$.

LEMMA 55. *Assume that A is hyperprojective from A. Let* $\mathrm{f}(x)$ *be totally defined, hyperprojective from A and such that for each x,* $\mathrm{f}(x) \subset H$. *Then*

$$\textit{supremum}_x\, \textit{infimum}_z\, \{|z| : \mathrm{f}(x) \to z\} < \kappa^A. \tag{14.3}$$

Proof. Towards a contradiction assume that $\mathrm{f}(x)$ satisfies the hypothesis but that the ordinal on the left of (14.3) is $\geqq \kappa^A$. Choose $\mathrm{g}(x)$ by Lemma 53 to be total, hyperprojective from A, such that $\mathrm{g}(x) \to z$ is hyperprojective from A and such that $g(x) \to z \Rightarrow (Ez')[z' \sim z \;\&\; f(x) \to z']$. Lemma 54 now implies that for each x, $g(x) \subset H$ and that

$$\{|z|: g(x) \to z\} \subset \{|z'|: f(x) \to z'\},$$

so that

$$\text{supremum}_x \ \text{infimum}_z \{|z|: g(x) \to z\} \geqq \kappa^A. \tag{14.4}$$

Thus we have

$$x \in H^A \Leftrightarrow x \in A^* \;\&\; (Ey)(Ez)[g(y) \to z \;\&\; p(x, z) \to 0]$$

which easily implies that H^A is hyperprojective from A, contradicting Lemma 49.

We call a partial ordering $x \preceq y$ *hyperprojective from* A if as a predicate (false when either x or y are not in the field of $\preceq$) it is hyperprojective from A. If $x \preceq y$ is a partial well-ordering we attach to each x in its field an ordinal $|x|^{\preceq}$ in the usual fashion:

$$\begin{aligned} |x|^{\preceq} &= 0 \textit{ if } x \textit{ is a minimal element,} \\ |x|^{\preceq} &= \textit{supremum } \{|y|^{\preceq}+1 : y \prec x\} \textit{ if } x \textit{ is not minimal,} \end{aligned} \tag{14.5}$$

where

$$x \prec y \Leftrightarrow x \preceq y \;\&\; \neg(y \preceq x). \tag{14.6}$$

Put

$$|\preceq| = \textit{supremum}\, \{|x|^{\preceq}: x \preceq x\}. \tag{14.7}$$

An ordinal ξ is *hyperprojective from* A if there exists a partial well-ordering $x \preceq y$ which is hyperprojective from A and such that $\xi = |\preceq|$.

THEOREM 9. (a) *Each ordinal $\xi < \lambda^A$ is hyperprojective from A.*
(b) *If A is hyperprojective from A, then λ^A is not hyperprojective from A.*

Proof. (a) is immediate by setting for each $z \in H^A$,

$$x \preceq_z y \Leftrightarrow [x \in A^* \;\&\; y \in A^* \;\&\; |x| \leqq |y| \;\&\; |y| < |z|]$$

and using Theorem 7.

To prove (b) by contradiction, suppose that $x \preceq y$ is hyperprojective from A and $|\preceq| = \lambda^A$. First put

$$\begin{aligned} h(f, y, x, z) &= 1 \quad \textit{if } \neg(y \prec x) \vee [y \prec x \;\&\; p(z, \{f\}_h(y)) \to 1, \\ &= 0 \quad \textit{if } y \prec x \;\&\; p(z, \{f\}_h(y)) \to 0. \end{aligned} \tag{14.8}$$

Using the recursion theorem, Lemma 41, we define a function $f(x)$ with index f such that

$$\begin{aligned} f(x) &= \nu z[z \in H^A] \textit{ if } \neg(x \preceq x) \vee (y)[y \preceq x \Rightarrow x \preceq y], \\ &= \nu z[z \in H^A \;\&\; \mathbf{E}(\lambda y h(f, y, x, z)) \to 1] \textit{ otherwise.} \end{aligned} \tag{14.9}$$

If $\neg(x \preceq x)$, then clearly $\mathrm{f}(x)$ is defined and in fact $\mathrm{f}(x)=H^A$. We prove by transfinite induction on $|x|^{\preceq}$ that

$$\text{(14.10)} \qquad \begin{gathered} \textit{if } x \preceq x, \textit{ then } \mathrm{f}(x) \textit{ is defined}, \mathrm{f}(x) \subset H^A \textit{ and} \\ \textit{infimum } \{|z|^A : \mathrm{f}(x) \to z\} = |x|^{\preceq}. \end{gathered}$$

If x is minimal then $\mathrm{f}(x)=H^A$, so (14.10) is trivially true. If x is not minimal, then by definition

$$\begin{aligned} \mathrm{f}(x) \to z &\Leftrightarrow z \in H^A \;\&\; (y)[\mathrm{h}(f, y, x, z) \to 1] \\ &\Leftrightarrow z \in H^A \;\&\; (y)[y \prec x \Rightarrow \mathrm{p}(z, \{f\}_{\mathrm{h}}(y)) \to 1]. \end{aligned}$$

Now $\mathrm{f}(y)=\{f\}_{\mathrm{h}}(y)$ is multiple-valued, so the last equivalence means

$$\begin{aligned} \mathrm{f}(x) \to z &\Leftrightarrow z \in H^A \;\&\; (y)[y \prec x \Rightarrow (Ez')[\mathrm{f}(y) \to z' \;\&\; |z'| < |z|]] \\ &\Leftrightarrow z \in H^A \;\&\; (y)[y \prec x \Rightarrow (Ez')[\mathrm{f}(y) \to z' \;\&\; |z'|^A < |z|^A]] \end{aligned}$$

where we have used the induction hypothesis to infer that if $y \prec x$ and $\mathrm{f}(y) \to z'$, then $z' \in H^A$ and also the obvious fact

$$\text{(14.11)} \qquad z, z' \in H^A \Rightarrow [|z'| < |z| \Leftrightarrow |z'|^A < |z|^A].$$

Now by induction hypothesis, for each $y \prec x$, *infimum*$_{z'}$ $\{|z'|^A : \mathrm{f}(y) \to z'\} = |y|^{\preceq}$. Hence

$$\mathrm{f}(x) \to z \Leftrightarrow z \in H^A \;\&\; (y)[y \prec x \Rightarrow |y|^{\preceq} < |z|^A] \Leftrightarrow z \in H^A \;\&\; |z|^A \geqq |x|^{\preceq}.$$

The hypothesis of the theorem implies then that $\mathrm{f}(x)$ is totally defined, that for each x, $\mathrm{f}(x) \subset H^A$ and that

$$\textit{supremum}_x \; \textit{infimum}_z \{|z|^A : \mathrm{f}(x) \to z\} = \lambda^A;$$

hence

$$\textit{supremum}_x \; \textit{infimum}_z \{|z| : \mathrm{f}(x) \to z\} = \kappa^A$$

contradicting Lemma 55.

REMARK 19. Let us call a partial well-ordering $\preceq$ *strongly hyperprojective from* A if it is hyperprojective from A and if its field is a subset of A^*. The proof of Theorem 9 shows that every ordinal $\xi < \lambda^A$ is realizable by a partial well-ordering which is strongly hyperprojective from A.

Suppose that $x \preceq y$ is strongly hyperprojective from A and $|\preceq| > \lambda^A$. Then there exists some x_0 in the field of $\preceq$ such that the partial ordering $\preceq$ restricted to $\{x : x \preceq x_0\}$ has ordinal λ^A. Since $x_0 \in A^*$, this new partial well-ordering is hyperprojective from A and hence it contradicts part (b) of Theorem 9. Thus λ^A is the *supremum of the ordinals realizable by partial well-orderings strongly hyperprojective from A*.

However it is not in general true that λ^A is the supremum of the ordinals realizable by partial well-orderings hyperprojective from A. Because if $x \in H^A$ and we consider the restriction of the partial ordering on H to $\{y : y \in H \;\&\; |y| < |x|\}$, this partial well-ordering is hyperprojective from A and has the ordinal $|x|$ associated with it which is in general greater than $|x|^A$ and perhaps greater than λ^A.

LEMMA 56. (a) *Suppose A^* can be well-ordered hyperprojectively from A. Then λ^A is the smallest ordinal not realizable by a well-ordering hyperprojective from A.*

(b) *If A is finite, then A^* can be well-ordered hyperprojectively from A.*

Proof. To prove (a) it is enough to show that every $\xi < \lambda^A$ can be realized by a well-ordering hyperprojective from A. Let $\preceq$ be the given well-ordering on A^*, put

$$x \in C \Leftrightarrow x \in A^* \;\&\; (y)[y \in A^* \;\&\; |y| = |x| \Rightarrow x \preceq y]$$

and then consider the partial ordering of H restricted to C.

(b) is immediate, since when A is finite, then A^* can be effectively enumerated without repetitions by a function in $PR(A)$.

15. **The length of the hyperprojective hierarchy.** For each $z \in H$, put

$$H_z(x) \Leftrightarrow |(x)_2| \leqq |z| \;\&\; |(x)_1| \leqq |(x)_2|. \tag{15.1}$$

The theorem below summarizes several results which we have essentially already proved.

We recall our blanket assumption that φ consists of single-valued total functions.

THEOREM 10. *Assume that A is hyperprojective from A.*

(a) *H^A is semi-hyperprojective from A, $H^A \subset A^*$.*

(b) *There is an absolutely hyperprojective function* $\mathrm{p}(x, z)$, *such that if $z \in H^A$, then $\lambda x\mathrm{p}(x, z)$ is total, single-valued and*

$$|x|^A \leqq |z|^A \Leftrightarrow \mathrm{p}(x, z) = 0,$$

i.e. in the partial well-ordering $\lambda xy|x|^A \leqq |y|^A$ on H^A, the set of predecessors of $z \in H^A$ is hyperprojective from A and uniformly in $z \in H^A$.

(c) *There is an absolutely hyperprojective function* $\mathrm{h}(x, z)$ *such that if $z \in H^A$, then $\lambda x\mathrm{h}(x, z)$ is the characteristic function of $H_z(x)$, i.e. $H_z(x)$ is hyperprojective from A, uniformly for $z \in H^A$.*

(d) *Each predicate $P(\boldsymbol{u})$ which is hyperprojective from A is search computable from A in some $H_z(x)$, $z \in H^A$.*

(e) *If $|z|^A \leqq |z'|^A$, then $H_z(x)$ is search computable from A in $H_{z'}(x)$.*

Proof. (a) is part (b) of Lemma 50. (b) comes from Theorem 7 and the observation that for x, z in H^A, $|x|^A \leqq |z|^A \Leftrightarrow |x| \leqq |z|$. (c) again is immediate from Theorem 7. (d) follows easily from Theorem 8. (e) is obvious from

$$|z|^A \leqq |z'|^A \Rightarrow H_z(x) \Leftrightarrow H_{z'}(\langle (x)_1, z\rangle) \;\&\; H_{z'}(x).$$

The assertions above seem to establish an adequate hierarchy for $HP(A)$, except for one thing: it is not in general true that if $|z|^A < |z'|^A$, then $H_{z'}$ is not search computable from A in H_z. Thus the problem is to show that the order type of

$$\{|z| : z \in H^A \ \&\ (w)[w \in H^A \ \&\ |w| < |z| \Rightarrow H_z \text{ is not search computable from } A \text{ in } H_w]\} \tag{15.2}$$

is what it should be, namely λ^A. In this section we show that any hierarchy for $HP(A)$ which satisfies the structural criteria of Theorem 10 must have length λ^A.

LEMMA 57. *Assume A is hyperprojective from A, let "f", "g" be variables over one-place p.m.v.'s. There is a functional* $\mathrm{comp}_1(\mathrm{f}, \mathrm{g})$, *hyperprojective from A, such that if* f *is single-valued, total and integer-valued and if* g *is single-valued, total, then* $\mathrm{comp}_1(\mathrm{f}, \mathrm{g})$ *is defined,* $\mathrm{comp}_1(\mathrm{f}, \mathrm{g}) = 0$ *or* 1 *and*

$$\mathrm{comp}_1(\mathrm{f}, \mathrm{g}) = 0 \Leftrightarrow \mathrm{f} \textit{ is search computable from } A \textit{ in } \mathrm{g}. \tag{15.3}$$

Proof. By the normal form Theorem 1,

$$\mathrm{f} \textit{ is search computable from } A \textit{ in } \mathrm{g}$$
$$\Leftrightarrow (Ee)[e \in A^* \ \&\ (x)(Ey)[T(\mathrm{g}, e, x, y) \ \&\ U(\mathrm{g}, e, x, y) \sim \mathrm{f}(x)]]$$

when the restrictions on f and g of the lemma are satisfied.

LEMMA 58. *Assume A is hyperprojective from A. There is a function* $\mathrm{comp}(f, g)$, *hyperprojective from A, such that if $\lambda x\{f\}_{\mathrm{h}}(x)$, $\lambda x\{g\}_{\mathrm{h}}(x)$ are single-valued and total and if $\lambda x\{f\}_{\mathrm{h}}(x)$ is integer-valued, then* $\mathrm{comp}(f, g)$ *is defined,* $\mathrm{comp}(f, g) = 0$ *or* 1 *and*

$$\mathrm{comp}(f, g) = 0 \Leftrightarrow \lambda x\{f\}_{\mathrm{h}}(x) \textit{ is search computable from } A \textit{ in } \lambda x\{g\}_{\mathrm{h}}(x). \tag{15.4}$$

Proof. Using the transitivity Lemma 43 and recalling that we have defined $\{h\}_{\mathrm{h}}(\boldsymbol{u})$ even for empty $\boldsymbol{u}$ we have

$$\begin{aligned}\mathrm{comp}_1(\lambda x\{f\}_{\mathrm{h}}(x), \lambda x\{g\}_{\mathrm{h}}(x)) &= \{\mathrm{tr}_{\mathrm{h}}(comp_1, f)\}_{\mathrm{h}}(\lambda x\{g\}_{\mathrm{h}}(\mathrm{x}))\\ &= \{\mathrm{tr}_{\mathrm{h}}(\mathrm{tr}_{\mathrm{h}}(comp_1, f), g)\}_{\mathrm{h}}(\ )\end{aligned}$$

where $comp_1$ is an index of $\mathrm{comp}_1(\mathrm{f}, \mathrm{g})$ from A, so we can put

$$\mathrm{comp}(f, g) = \{\mathrm{tr}_{\mathrm{h}}(\mathrm{tr}_{\mathrm{h}}(comp_1, f), g)\}_{\mathrm{h}}(\ ).$$

THEOREM 11. *Assume that A is hyperprojective from A. Let G be a set, $\preceq$ a partial well-ordering on G and for each $z \in G$ let $G_z(x)$ be a predicate, so that the following conditions are satisfied*:

(a) *G is semi-hyperprojective from A, $G \subset A^*$.*

(b) *There is a function* $\mathrm{p}_1(x, z)$, *hyperprojective from A, such that if $z \in G$, then* $\lambda x \mathrm{p}_1(x, z)$ *is completely defined, single-valued and*

$$|x|^{\preceq} \leqq |z|^{\preceq} \Leftrightarrow \mathrm{p}_1(x, z) = 0.$$

(c) *There is a function* $h_1(x, z)$, *hyperprojective from* A, *such that if* $z \in G$, *then* $\lambda x h_1(x, z)$ *is the characteristic function of* $G_z(x)$.

(d) *Each predicate* $P(\boldsymbol{u})$ *which is hyperprojective from* A *is search computable from* A *in some* $G_z(x)$, $z \in G$.

Then the ordinal $|\preceq|$ *associated with the partial well-ordering* $\preceq$ *is* λ^A.

Proof. For each $z \in G$, the partial well-ordering

$$x \preceq_z y \Leftrightarrow |y|^{\preceq} < |z|^{\preceq} \;\&\; |x|^{\preceq} \leqq |y|^{\preceq}$$

is strongly hyperprojective from A with ordinal $|z|^{\preceq}$, hence by Remark 19, $|z|^{\preceq} < \lambda^A$, so that $|\preceq| \leqq \lambda^A$, We shall derive a contradiction from the assumption

(15.5) $$|\preceq| = |w_0|^A, \quad w_0 \in H^A.$$

Let us first show that there is no $y \in G$ so that

(15.6) *for each* $z \in H^A$, *there is a* $v \in G$, $|v|^{\preceq} < |y|^{\preceq}$, *such that* H_z *is search computable from* A *in* G_v;

because if (15.6) were true for some $y \in G$, then every set hyperprojective from A would be search computable from A in the predicate $P(x)$ defined by

$$P(x) \Leftrightarrow |(x)_1|^{\preceq} < |y|^{\preceq} \;\&\; G_{(x)_1}((x)_0),$$

which by assumptions (b) and (c) of the theorem is hyperprojective from A. However, it is very easy to show that there are always predicates hyperprojective from A which are not search computable in P, e.g. take $(Ey)T(\chi_P, x, x, y)$ and use Lemma 44 and Theorem 4.

Put

(15.7) $$g(y) = \nu z[z \in H^A \;\&\; (v)[|v|^{\preceq} < |y|^{\preceq} \Rightarrow H_z \textit{ is not search computable from } A \textit{ in } G_v]].$$

Assumptions (b) and (c) of the theorem and Lemma 58 imply that $g(y)$ is hyperprojective from A. The fact that (15.6) cannot hold for any $y \in G$, implies that if $y \in G$, then $g(y)$ is defined, by definition $g(y) \subset H^A$, and clearly

$$|y|^{\preceq} \leqq |y'|^{\preceq} \Rightarrow g(y') \subset g(y).$$

Finally, assumption (d) of the theorem and part (e) of Theorem 10 imply that

(15.8) $$\textit{supremum}_{y \in G}\, \textit{infimum}_z\, [|z| : g(y) \to z] = \kappa^A.$$

We now define a function $f(x)$, using the recursion theorem, Lemma 41 and multiple-valued definition by cases, Lemma 30, so that the following conditions are satisfied:

(15.9)
$$\begin{aligned} f(x) &= \nu z[z \in G] \textit{ if } \neg[x \in A^* \;\&\; |x|^A < |w_0|^A],\\ &= \nu z[z \in G] \textit{ if } x = \langle 1, 0\rangle,\\ &= \nu z[z \in G \;\&\; (y)[|y|^A < |x|^A \Rightarrow |f(y)|^{\preceq} < |z|^{\preceq}]]\\ &\qquad \textit{if } x \in A^* \;\&\; |x|^A < |w_0|^A \;\&\; x \neq \langle 1, 0\rangle. \end{aligned}$$

A careful statement of the third case would of course involve **E**; the multiple-valued interpretation of that case is

$$\mathrm{f}(x) \to z \Leftrightarrow z \in G \;\&\; (y)[|y|^A < |x|^A \Rightarrow (Ez')[\mathrm{f}(y) \to z' \;\&\; |z'|^{\preceq} < |z|^{\preceq}]].$$

It is immediate that, if $\neg[x \in A^* \;\&\; |x|^A < |w_0|^A]$, then $\mathrm{f}(x) = G$. It is also easy to show by transfinite induction on $|x|^A$, that if $x \in A^* \;\&\; |x|^A < |w_0|^A$, then

$$\text{(15.10)} \qquad \mathit{infimum}_z\, [|z|^{\preceq}\colon \mathrm{f}(x) \to z] = |x|^A.$$

Now consider the function

$$\text{(15.11)} \qquad \mathrm{h}(x) = \mathrm{g}(\mathrm{f}(x)),$$

where the composition is interpreted as composition of multiple-valued functions. It is hyperprojective from A, completely defined, for each x, $\mathrm{h}(x) \subset H$ (by (15.9) and (15.7)) and by (15.10) and (15.8),

$$\mathit{supremum}_x\; \mathit{infimum}_z\, \{|z|\colon \mathrm{h}(x) \to z\} = \kappa^A,$$

contradicting Lemma 55.

REMARK 20. We can prove (15.2) from Theorem 11, by taking

$$G = \{z \in H^A\colon (w)[|w| < |z| \Rightarrow H_z \textit{ is not search computable from } A \textit{ in } H_w\}$$

and taking $\preceq$ to be the restriction of the ordering on H to G.

16. **The first recursion theorem for hyperprojective functions.** Lemma 53 was very useful in proving Lemma 55, which again was the main tool for proving Theorems 9 and 11. We give here a name to the functions whose existence was shown in Lemma 53.

A p.m.v. function $\mathrm{f}(\boldsymbol{u})$ is *normal* (from A) if the predicate $\mathrm{f}(\boldsymbol{u}) \to z$ is hyperprojective (from A). It is clear that normal functions are hyperprojective, since $\mathrm{f}(\boldsymbol{u}) = \nu z[\mathrm{f}(\boldsymbol{u}) \to z]$.

Put

$$\text{(16.1)} \qquad \mathrm{f} \subset_{\sim} \mathrm{g} \Leftrightarrow (\boldsymbol{u})(z)[\mathrm{f}(\boldsymbol{u}) \to z \Rightarrow (Ez')[\mathrm{g}(\boldsymbol{u}) \to z' \;\&\; z \sim z']].$$

It is immediate that if $x = y$ is hyperprojective, then single-valued total hyperprojective functions are normal and $\mathrm{f} \subset_{\sim} \mathrm{g}$ coincides with $\mathrm{f} \subset \mathrm{g}$.

LEMMA 59. *If* $\{f\}_{\mathrm{h}}(\chi, \boldsymbol{u}) \to z$ *and* $\chi \subset_{\sim} \chi'$, *then for some* $z' \sim z$, $\{f\}_{\mathrm{h}}(\chi', \boldsymbol{u}) \to z'$. *Hence, if* $\mathrm{f}(\chi, \boldsymbol{u})$ *is a* p.m.v. *hyperprojective functional and* $\chi \subset_{\sim} \chi'$, *then*

$$\mathrm{f}(\chi, \boldsymbol{u}) \subset_{\sim} \mathrm{f}(\chi', \boldsymbol{u}).$$

Proof is by induction on C0′–C10′.

Case C0. Subcase (a). $\{f\}(\chi, t_1, \ldots, t_{n_0}, \boldsymbol{x}) = \chi(t_1, \ldots, t_{n_0})$. If $\{f\}_{\mathrm{h}}(\chi, t_1, \ldots, t_{n_0}, \boldsymbol{x}) \to z$, then $\chi(t_1, \ldots, t_{n_0}) \to z$, hence $\chi'(t_1, \ldots, t_{n_0}) \to z' \sim z$ by assumption, hence $\{f\}_{\mathrm{h}}(\chi', t_1, \ldots, t_{n_0}, \boldsymbol{x}) \to z'$.

Case C0. Subcase (b) and C1–C4 are trivial.

Case C5. $\{f\}(\chi, \boldsymbol{x})=\{g\}(\chi, \{h\}(\chi, \boldsymbol{x}), \boldsymbol{x})$. If $\{f\}_{\mathrm{h}}(\chi, \boldsymbol{x}) \to z$, then for some u $\{h\}_{\mathrm{h}}(\chi, \boldsymbol{x}) \to u$ and $\{g\}_{\mathrm{h}}(\chi, u, \boldsymbol{x}) \to z$, hence by ind. hyp. there exist u', z' so that $\{h\}_{\mathrm{h}}(\chi', \boldsymbol{x}) \to u'$, $\{g\}_{\mathrm{h}}(\chi', u, \boldsymbol{x}) \to z'$ and $u' \sim u$, $z' \sim z$. Now by Lemma 45,

$$\{g\}_{\mathrm{h}}(\chi', u', \boldsymbol{x}) \to z''$$

for some $z'' \sim z' \sim z$, hence $\{f\}_{\mathrm{h}}(\chi', \boldsymbol{x}) \to z'' \sim z$.

Case C6 is handled like Case C5 and Cases C7, C8, C9 are immediate.

Case C10. $\{f\}(\chi, \boldsymbol{x})=\mathbf{E}(\lambda y\{g\}(\chi, y, \boldsymbol{x}))$. Assume that $\{f\}_{\mathrm{h}}(\chi, \boldsymbol{x}) \to 1$; then for each y, $\{g\}_{\mathrm{h}}(\chi, y, \boldsymbol{x}) \to u_y \neq 0$, so by ind. hyp. for each y there is a $u'_y \sim u_y$ (hence $u'_y \neq 0$) such that $\{g\}_{\mathrm{h}}(\chi', y, \boldsymbol{x}) \to u'_y$ which immediately implies $\{f\}_{\mathrm{h}}(\chi', \boldsymbol{x}) \to 1$. Similarly for the case $\{f\}_{\mathrm{h}}(\chi, \boldsymbol{x}) \to 0$.

We recall our blanket assumption that $\boldsymbol{\varphi}$ consists of total, single-valued functions.

LEMMA 60. *If $\{f\}_{\mathrm{h}}(\chi, \boldsymbol{u}) \to z$, then there exists a $\chi' \subset_{\sim} \chi$ and a $z' \sim z$ such that χ' is normal in χ and $\{f\}_{\mathrm{h}}(\chi', \boldsymbol{u}) \to z'$.*

Proof is again by induction on C0′–C10′.

Case C0. Subcase (a). $\{f\}(\chi, t_1, \ldots, t_{n_0}, \boldsymbol{x})=\chi(t_1, \ldots, t_{n_0})$. Put

$$\chi'(t'_1, \ldots, t_{n_0}) \to z' \Leftrightarrow t'_1 \sim t_1 \ \& \cdots \& \ t'_{n_0} \sim t_{n_0} \ \& \ z' \sim z,$$

where we assume that $\{f\}_{\mathrm{h}}(\chi, t_1, \ldots, t_{n_0}, \boldsymbol{x}) \to z$. Clearly $\chi' \subset_{\sim} \chi$, χ' is normal and $\{f\}_{\mathrm{h}}(\chi', t_1, \ldots, t_{n_0}, \boldsymbol{x}) \to z$.

Case C0. Subcase (b) and Cases C1–C4. Take χ' to be the totally undefined function.

Case C5. $\{f\}(\chi, \boldsymbol{x})=\{g\}(\chi, \{h\}(\chi, \boldsymbol{x}), \boldsymbol{x})$. If $\{f\}_{\mathrm{h}}(\chi, \boldsymbol{x}) \to z$, then for some u, $\{h\}_{\mathrm{h}}(\chi, \boldsymbol{x}) \to u$, $\{g\}_{\mathrm{h}}(\chi, u, \boldsymbol{x}) \to z$, so by ind. hyp. there exist $u' \sim u$, $z' \sim z$, $\chi' \subset_{\sim} \chi$, $\chi'' \subset_{\sim} \chi$, both χ', χ'' normal in χ, such that $\{h\}_{\mathrm{h}}(\chi', \boldsymbol{x}) \to u'$, $\{g\}_{\mathrm{h}}(\chi'', u, \boldsymbol{x}) \to z'$. Put

$$\chi'''(t_1, \ldots, t_{n_0}) \to z \Leftrightarrow [\chi'(t_1, \ldots, t_{n_0}) \to z] \vee [\chi''(t_1, \ldots, t_{n_0}) \to z]. \tag{16.2}$$

Clearly χ''' is normal in χ, $\chi''' \subset_{\sim} \chi$ and since $\chi' \subset_{\sim} \chi'''$, $\chi'' \subset_{\sim} \chi'''$, we have $\{h\}_{\mathrm{h}}(\chi''', \boldsymbol{x}) \to u''$, $\{g\}_{\mathrm{h}}(\chi''', u, \boldsymbol{x}) \to z''$, by Lemma 59, where $u'' \sim u' \sim u$, $z'' \sim z' \sim z$. The result follows immediately by an application of Lemma 45.

Cases C6–C9 are handled similarly.

Case C10. $\{f\}(\chi, \boldsymbol{x})=\mathbf{E}(\lambda y\{g\}(\chi, y, \boldsymbol{x}))$. Suppose $\{f\}_{\mathrm{h}}(\chi, \boldsymbol{x}) \to 1$. By ind. hyp. then for each y there is a $\chi_y \subset_{\sim} \chi$, χ_y normal in χ, and a $u_y \neq 0$ such that $\{g\}_{\mathrm{h}}(\chi_y, y, \boldsymbol{x}) \to u_y$. For each y, since χ_y is normal in χ, there exists an m_y such that

$$\chi_y(t_1, \ldots, t_{n_0}) \to z \Leftrightarrow \mathrm{eh}(m_y, t_1, \ldots, t_{n_0}, z) \in H(\chi),$$

by Theorem 8. (Here $H(\chi)$ is H relative to the list χ, $\boldsymbol{\varphi}$. Similarly for $x \in H(\chi)$ we put $|x|(\chi)$ for the ordinal associated with x by (12.1).)

Put

$$\begin{aligned} \mathrm{f}(y) = \nu v[(v)_0 \in H(\chi) \ \& \ (Ez')[\{g\}_{\mathrm{h}}(\chi^v, y, \boldsymbol{x}) \to z' \ \& \ z' \sim (v)_1] \\ \& \ (v)_1 \neq 0 \ \& \ \chi^v \subset_{\sim} \chi], \end{aligned} \tag{16.3}$$

where

(16.4) $\quad \chi^v(t_1, \ldots, t_{n_0}) \to z \Leftrightarrow |\mathrm{eh}((v)_2, t_1, \ldots, t_{n_0}, z)|(\chi) \leqq |(v)_0|(\chi).$

Since an index of the hyperprojective function χ^v can be easily computed from v, we can use the transitivity Lemma 43 to see that $\mathrm{f}(y)$ is hyperprojective in χ. Since $\mathrm{f}(y)$ is completely defined, Lemma 53 applies and there is a normal (in χ) total function $\mathrm{f}'(y)$ such that $\mathrm{f}' \subset_{\sim} \mathrm{f}$. Put

(16.5)
$$\begin{aligned} &\chi'(t_1, \ldots, t_{n_0}) \to z \\ &\quad \Leftrightarrow (Ey)(Ev')[\mathrm{f}'(y) \to v' \;\&\; |\mathrm{eh}((v')_2, t_1, \ldots, t_{n_0}, z)|(\chi) \leqq |(v')_0|(\chi)]. \end{aligned}$$

It is immediate that χ' is normal in χ. The predicate in brackets in (16.3) is semi-hyperprojective in χ, hence by Lemma 45, $\mathrm{f}(y) \to v$ and $v' \sim v \Rightarrow \mathrm{f}(y) \to v'$; hence $\mathrm{f}' \subset \mathrm{f}$, so that if $\mathrm{f}'(y) \to v$, then $\mathrm{f}(y) \to v$ and $\chi^v \subset_{\sim} \chi$, hence $\chi' \subset_{\sim} \chi$. Moreover, for each y there is a v such that $\mathrm{f}'(y) \to v$, $(v)_1 \neq 0$ and $\{g\}_{\mathrm{h}}(\chi^v, y, \boldsymbol{x}) \to (v)_1$, so that $\{g\}_{\mathrm{h}}(\chi', y, \boldsymbol{x}) \to (v)_1$ and $\{f\}_{\mathrm{h}}(\chi', x) \to 1$.

The argument is similar on the assumption $\{f\}_{\mathrm{h}}(\chi, \boldsymbol{x}) \to 0$.

THEOREM 12. *Let* $\mathrm{f}(\chi, \boldsymbol{x})$ *be a p.m.v. functional which is hyperprojective from A. There exists a p.m.v. function χ, hyperprojective from A, such that*

(16.6) $\quad \mathrm{f}(\chi, \boldsymbol{x}) =_{\sim} \chi(\boldsymbol{x})$, *all* $\boldsymbol{x}$ (*i.e.* $\mathrm{f}(\chi, \boldsymbol{x}) \subset_{\sim} \chi(\boldsymbol{x})$ *and* $\chi(\boldsymbol{x}) \subset_{\sim} \mathrm{f}(\chi, \boldsymbol{x})$)

and

(16.7) $\quad$ *if* $\mathrm{f}(\chi', \boldsymbol{x}) =_{\sim} \chi'(\boldsymbol{x})$, *all* $\boldsymbol{x}$, *then* $\chi \subset_{\sim} \chi'$.

(*The first recursion theorem.*)

Proof is similar to that of Theorem 2. To each $z \in H$ we assign a function

(16.8)
$$\chi_{\langle 1,0\rangle} = \varnothing, \quad \chi_z(\boldsymbol{x}) = \mathrm{f}\Big(\bigcup_{|z'|<|z|} \chi_{z'}, \boldsymbol{x}\Big)$$

and we put

(16.9)
$$\chi(\boldsymbol{x}) = \mathit{limit}_{z\in H}\, \chi_{z'}(\boldsymbol{x}).$$

Clearly χ_z depends only on the ordinal $|z|$.

Using Lemma 59 and transfinite induction we easily obtain

(16.10)
$$|z'| \leqq |z| \Rightarrow \chi_{z'} \subset_{\sim} \chi_z.$$

Again, if $\mathrm{f}(\chi', \boldsymbol{x}) =_{\sim} \chi'(\boldsymbol{x})$, all $\boldsymbol{x}$, then we easily show by transfinite induction that $\chi_z \subset_{\sim} \chi'$, so $\chi \subset_{\sim} \chi'$ and we have proved (16.7).

Consider the function

(16.11)
$$\begin{aligned} \mathrm{g}(p, z, \boldsymbol{x}) &= \bigcup_{|z'|<|z|} \{\{p\}_{\mathrm{pr}}(z')\}_{\mathrm{h}}(\boldsymbol{x}) \\ &= \{\{p\}_{\mathrm{pr}}(\nu z'[z \in H \;\&\; |z'| < |z|])\}_{\mathrm{h}}(\boldsymbol{x}); \end{aligned}$$

it is clearly absolutely hyperprojective, so let $g \in 0^*$ be an index of it. Using the recursion theorem for primitive computable functions Lemma 15, choose p so that

$$\{p\}_{\mathrm{pr}}(z) = \mathrm{tr}_{\mathrm{h}}(f, S^2(g, p, z)), \tag{16.12}$$

where f is an index of f and $\mathrm{tr}_{\mathrm{h}}(f, c)$ is the transitivity function of Lemma 43. Then for $z \in H$,

$$\chi_z(\boldsymbol{x}) = \{\{p\}_{\mathrm{pr}}(z)\}_{\mathrm{h}}(\boldsymbol{x}) \tag{16.13}$$

by the following transfinite induction:

$$\begin{aligned} \chi_z(\boldsymbol{x}) &= \{f\}_{\mathrm{h}}\Big(\bigcup_{|z'|<|z|} \chi_{z'}, \boldsymbol{x}\Big) \\ &= \{f\}_{\mathrm{h}}\Big(\lambda \boldsymbol{x} \bigcup_{|z'|<|z|} \{\{p\}_{\mathrm{pr}}(z)\}_{\mathrm{h}}(\boldsymbol{x}), \boldsymbol{x}\Big) \quad \text{by ind. hyp.} \\ &= \{f\}_{\mathrm{h}}(\lambda \boldsymbol{x}\{S^2(g, p, z)\}_{\mathrm{h}}(\boldsymbol{x}), \boldsymbol{x}) = \{\mathrm{tr}_{\mathrm{h}}(f, S^2(g, p, z))\}_{\mathrm{h}}(\boldsymbol{x}) \\ &= \{\{p\}_{\mathrm{pr}}(z)\}_{\mathrm{h}}(\boldsymbol{x}). \end{aligned}$$

Hence $\chi(\boldsymbol{x})$ is hyperprojective from A, since

$$\chi(\boldsymbol{x}) = \{\{p\}_{\mathrm{pr}}(\nu z[z \in H])\}_{\mathrm{h}}(\boldsymbol{x}). \tag{16.14}$$

It remains to show (16.6). An easy transfinite induction shows that for each $z \in H$, $\chi_z(\boldsymbol{x}) \subset_\sim \mathrm{f}(\chi, \boldsymbol{x})$, hence $\chi(\boldsymbol{x}) \subset_\sim \mathrm{f}(\chi, \boldsymbol{x})$. To show also $\mathrm{f}(\chi, \boldsymbol{x}) \subset_\sim \chi(\boldsymbol{x})$, assume that $\mathrm{f}(\chi, \boldsymbol{x}) \to w$, hence by Lemma 60 and since χ is hyperprojective there is a normal $\chi' \subset_\sim \chi$ such that $\mathrm{f}(\chi', \boldsymbol{x}) \to w$; we shall show that for some $z \in H$, $\chi' \subset_\sim \chi_z$, so that $\chi_z(\boldsymbol{x}) \to w'$, $w' \sim w$, and hence $\chi(\boldsymbol{x}) \to w'$ as required. Put

$$\begin{aligned} \mathrm{h}(\boldsymbol{x}, z) &= \langle 1, 0 \rangle \quad \text{if } \neg[\chi'(\boldsymbol{x}) \to z], \\ &= \nu u[u \in H \ \& \ \chi_u(\boldsymbol{x}) \sim z] \quad \text{if } \chi'(\boldsymbol{x}) \to z. \end{aligned} \tag{16.15}$$

By our multiple-valued interpretation of composition,

$$\chi_u(\boldsymbol{x}) \sim z \Leftrightarrow (Ez')[\chi_u(\boldsymbol{x}) \to z' \ \& \ z' \sim z],$$

and this predicate is semi-hyperprojective from A, since $u \sim v$ is absolutely hyperprojective and $\chi_u(x)$ is hyperprojective from A uniformly for $u \in H$ by (16.13). Thus $\mathrm{h}(\boldsymbol{x}, z)$ is hyperprojective from A and since $\chi' \subset_\sim \chi$, $\mathrm{h}(\boldsymbol{x}, z)$ is completely defined, $\mathrm{h}(\boldsymbol{x}, z) \subset H$. By Lemma 55 then there is a $z_0 \in H^A$ so that

$$\textit{supremum}_z \ \textit{infimum}_u \{|u| : \mathrm{h}(\boldsymbol{x}, z) \to u\} \leq |z_0|;$$

it follows immediately that $\chi' \subset_\sim \chi_{z_0}$.

The last part of the proof shows in fact that

$$\chi(\boldsymbol{x}) = \textit{limit}_{z \in H^A} \chi_z(\boldsymbol{x}). \tag{16.16}$$

REMARK 21. Since for integers i, j, $i = j \Leftrightarrow i \sim j$, Theorem 12 implies immediately that *if* $\mathrm{f}(\chi, \boldsymbol{x})$ *is integer-valued hyperprojective from* A, *then the least p.m.v.* χ *that satisfies*

$$(\boldsymbol{x})[\mathrm{f}(\chi, \boldsymbol{x}) = \chi(\boldsymbol{x})] \tag{16.17}$$

is hyperprojective from A.

Let $\mathbf{F}$ be a function on subsets of B^* to subsets of B^*; we call $\mathbf{F}$ *monotone*, if

$$C \subseteq D \Rightarrow \mathbf{F}(C) \subset \mathbf{F}(D). \tag{16.18}$$

For any monotone $\mathbf{F}$, put

$$C_0(\mathbf{F}) = \varnothing, \quad C_\xi(\mathbf{F}) = \mathbf{F}\Big(\bigcup_{\eta<\xi} C_\eta(\mathbf{F})\Big), \quad C(\mathbf{F}) = \bigcup_{\xi} C_\xi(\mathbf{F}); \tag{16.19}$$

it is evident from cardinality considerations that for some ordinal ξ, $C_\xi(\mathbf{F}) = C(\mathbf{F})$ and that $C(\mathbf{F})$ is the least set that satisfies

$$\mathbf{F}(C) = C. \tag{16.20}$$

Let us call $\mathbf{F}$ hyperprojective from A if the (partial, single-valued) function $\mathrm{f}(\chi, x)$ defined by

$$\mathrm{f}(\chi, x) \to 0 \Leftrightarrow x \in \mathbf{F}(\{y : \chi(y) \to 0\}) \tag{16.21}$$

is hyperprojective from A. In the notation of the proof of the theorem, we then have for each $z \in H$,

$$\chi_z(x) \to 0 \Leftrightarrow x \in C_{|z|}(\mathbf{F}) \tag{16.22}$$

and the partial function

$$\chi(x) \to 0 \Leftrightarrow x \in C(\mathbf{F}) \tag{16.23}$$

is the least solution of (16.17). Thus, *if* $\mathbf{F}$ *is monotone and hyperprojective from* A, *then* $C(\mathbf{F})$ *is semi-hyperprojective from* A *and* $C(\mathbf{F}) = C_\xi(\mathbf{F})$ *for some* $\xi \leqq \kappa$.

Put

$$\begin{aligned} x \in \mathbf{F}(C) \Leftrightarrow x = \langle 1, 0\rangle \vee [x &= \langle 2, e\rangle \;\&\; e \in PRI_1 \;\&\; (v)[e_v \in C]] \\ \vee [x &= \langle 3, e\rangle \;\&\; e \in PRI_1 \;\&\; (Ev)[e_v \in C]], \end{aligned} \tag{16.24}$$

where $e = (x)_2$. It is immediate that $\mathbf{F}$ is monotone and absolutely hyperprojective and that $C(\mathbf{F}) = H$. Thus *a relation* $P(\boldsymbol{u})$ *is in* $sHP(A)$ *if and only if*

$$P(\boldsymbol{u}) \Leftrightarrow \mathrm{g}(\boldsymbol{u}) \in C(\mathbf{F}) \tag{16.25}$$

for some monotone absolutely hyperprojective $\mathbf{F}$ *and some* $\mathrm{g}(\boldsymbol{u})$, *primitive computable from* A.

17. **Representation theorems.** The chief problem of [5] was to find explicit forms for the predicate $x \in O$ in a suitable language and similarly, the main results of [6] are the explicit forms that Kleene obtains for $\{e\}(\mathfrak{a}) \simeq z$. We devote the last three sections of this paper to such representation theorem for H and hence for all hyperprojective predicates.

The letters "S", "T", "U" will be used as variables for one-place fully defined functions on B^* with range in $\{0, 1\}$, i.e. characteristic functions for sets. A predicate $R(S, T, \boldsymbol{u})$ is in a class of predicates, e.g. $PR(A)$ or $HP(A)$, if there is a predicate $R'(\chi, \psi, \boldsymbol{u})$ in this class which coincides with R when χ, ψ are restricted to be characteristic functions.

We shall be using the restricted quantifiers $(ES)_{HP(A)}$, $(S)_{HP(A)}$,

$$(ES)_{HP(A)}R(S) \Leftrightarrow (ES)[S \in HP(A) \;\&\; R(S)], \tag{17.1}$$

$$(S)_{HP(A)}R(S) \Leftrightarrow (S)[S \in HP(A) \Rightarrow R(S)]. \tag{17.2}$$

We recall our blanket assumption that $\boldsymbol{\varphi}$ *consists of total, single-valued functions.* Put

$$\begin{aligned} g(z, x) &= 0 \quad \text{if } (z)_1 \in H \;\&\; (z)_2 \in PRI_1 \;\&\; |\{(z)_2\}_{\mathrm{pr}}(x)| \leqq |(z)_1|, \\ &= 1 \quad \text{if } (z)_1 \in H \;\&\; \neg[(z)_2 \in PRI_1 \;\&\; |\{(z)_2\}_{\mathrm{pr}}(x)| \leqq |(z)_1|], \end{aligned} \tag{17.3}$$

$$g_z(x) = g(z, x). \tag{17.4}$$

It is clear that the function $g(z, x)$ is absolutely hyperprojective. Using Theorem 8, it is easy to see that as $(z)_1$ varies over H^A and $(z)_2$ varies over A^*, $g_z(x)$ varies over all characteristic functions which are hyperprojective from A.

LEMMA 61. *Assume A is hyperprojective from A, let $R(x, S)$ be semi-hyperprojective from A.*

$$(x)(ES)R(x, S) \Leftrightarrow (ES)(x)(Ey)R(x, \lambda t S(\langle y, t\rangle)), \tag{17.5}$$

$$(x)(ES)_{HP(A)}R(x, S) \Leftrightarrow (ES)_{HP(A)}(x)(Ey)R(x, \lambda t S(\langle y, t\rangle)). \tag{17.6}$$

Proofs of (17.5) and of the right-to-left direction of (17.6) are trivial. To prove (17.6) from the left to the right we compute:

$$\begin{aligned} (x)(ES)_{HP(A)}R(x, S) &\Rightarrow (x)(Ez)[(z)_1 \in H^A \;\&\; (z)_2 \in A^* \;\&\; R(x, g_z)] \\ &\Rightarrow (Ez_0)[z_0 \in H^A \;\&\; (x)(Ez)[(z)_1 \in H^A \;\&\; |(z)_1| \leqq |z_0| \\ &\qquad\qquad \& \; (z)_2 \in A^* \;\&\; R(x, g_z)]], \end{aligned}$$

where we have used Lemma 55. We then obtain the right-hand side if we put

$$\begin{aligned} S(u) &= g((u)_1, (u)_2) \quad \text{if Seq}\,(u) \;\&\; (u)_0 = 2 \;\&\; |(u)_{1,1}| \leqq |z_0| \;\&\; (u)_{1,2} \in A^*, \\ &= 0 \qquad\qquad\qquad \text{otherwise.} \end{aligned}$$

The duals of (17.5) and (17.6) are

$$(Ex)(S)R(x, S) \Leftrightarrow (S)(Ex)(y)R(x, \lambda t S(\langle y, t\rangle)), \tag{17.7}$$

$$(Ex)(S)_{HP(A)}R(x, S) \Leftrightarrow (S)_{HP(A)}(Ex)(y)R(x, \lambda t S(\langle y, t\rangle)); \tag{17.8}$$

they hold in case R is the negative of a predicate semi-hyperprojective from A. All four equivalences (17.5)–(17.8) hold if R is hyperprojective from A.

LEMMA 62. *Let $R(x_1,\ldots,x_n)$ be semi-hyperprojective from A, let Q_i be E or $\forall$ for $i=1,\ldots,n$. Then*

$$(Q_1x_1)\cdots(Q_nx_n)R(x_1,\ldots,x_n) \Leftrightarrow (ES)(u)(Ev)R'(S,u,v), \tag{17.9}$$

$$(Q_1x_1)\cdots(Q_nx_n)R(x_1,\ldots,x_n) \Leftrightarrow (ES)_{HP(A)}(u)(Ev)R'(S,u,v) \tag{17.10}$$

where R' is a positive propositional combination of absolutely primitive computable predicates and predicates obtained from R by substitution of absolutely primitive computable functions for the variables of R, so that if R is in any of the classes $PR(A)$, $SC(A)$, $\pi_n^0(A)$, $\sigma_n^0(A)$, $HP(A)$, so is R'.

Proof. We first contract adjacent like quantifiers in the prefix of R using the trivial equivalence $(x)(y)Q(x,y) \Leftrightarrow (u)Q((u)_1,(u)_2)$, so that we may assume that $(Q_1x_1)\cdots(Q_nx_n)R$ is of the form

$$(x_1)(Ey_1)\cdots(x_n)(Ey_n)R(x_1,y_1,\ldots,x_n,y_n).$$

The construction of R' is by induction on n. Suppose then that $n>1$, put $P(x_1,y_1) \Leftrightarrow (x_2)(Ey_2)\cdots(x_n)(Ey_n)R$, so that the given expression is $(x_1)(Ey_1)P(x_1,y_1)$. Now the equivalences

$$\begin{aligned}(x_1)(Ey_1)P(x_1,y_1) \Leftrightarrow (ES)[(x_1)(Ey_1)&S(\langle x_1,y_1\rangle)=0\\ &\&\ (x_1)(y_1)[S(\langle x_1,y_1\rangle)=0 \Rightarrow P(x_1,y_1)]],\end{aligned} \tag{17.11}$$

$$\begin{aligned}(x_1)(Ey_1)P(x_1,y_1) \Leftrightarrow (ES)_{HP(A)}[(x_1)(Ey_1)&S(\langle x_1,y_1\rangle)=0\\ &\&\ (x_1)(y_1)[S(\langle x_1,y_1\rangle)=0 \Rightarrow P(x_1,y_1)]]\end{aligned} \tag{17.12}$$

are trivial, except perhaps for the left-to-right implication in (17.12). To prove this, assume $(x_1)(Ey_1)P(x_1,y_1)$, put $g(x_1)=\nu y_1P(x_1,y_1)$ and by the normal branch lemma, Lemma 53, let $g' \subset_{\sim} g$ be such that $g'(x_1)\to y_1$ is hyperprojective from A. Put

$$\begin{aligned}S(u) &= 0 \quad \text{if } g'((u)_1)\to(u)_2,\\ &= 1 \quad \text{otherwise}\end{aligned}$$

and then verify that S satisfies the right-hand side of (17.12). (If $S(\langle x_1,y_1\rangle)=0$, then for some $y_1'\sim y_1$, $g(x_1)\to y_1'$, so $P(x_1,y_1')$, hence $P(x_1,y_1)$ by Lemma 45 since $P(x_1,y_1)$ is semi-hyperprojective.)

Now the expression $(x_1)(y_1)[S(\langle x_1,y_1\rangle)=0 \Rightarrow P(x_1,y_1)]$ after trivial lower predicate calculus transformations and contractions of variables can be put in prenex form with $n-1$ alterations of quantifiers, so by ind. hyp. it can be put in the form

$$(ET)(u)(Ev)P'(T,u,v) \Leftrightarrow (ET)_{HP(A)}(u)(Ev)P'(T,u,v).$$

Hence

$$\begin{aligned}(x_1)(Ey_1)P(x_1,y_1) &\Leftrightarrow (ES)[(x_1)(Ey_1)S(\langle x_1,y_1\rangle)=0\ \&\ (ET)(u)(Ev)P'(T,u,v)]\\ &\Leftrightarrow (ES)_{HP(A)}[(x_1)(Ey_1)S(\langle x_1,y_1\rangle)=0\\ &\qquad\qquad \&\ (ET)_{HP(A)}(u)(Ev)P'(T,u,v)]\end{aligned}$$

which easily implies the result by a careful bringing of the quantifiers to the front and contracting of adjacent like quantifiers.

THEOREM 13. *There is an absolutely primitive computable predicate* $R(S, u, v, x)$, *such that*

$$x \in H \Leftrightarrow (S)(Eu)(v)R(S, u, v, x). \tag{17.13}$$

(*The first representation theorem.*)

Proof. This is a simple analysis from the outside of the inductive definition of H. Put

$$\begin{aligned}\mathrm{SAT}(S) \Leftrightarrow S(\langle 1, 0\rangle) = 0 \;\&\; (e)[e \in PRI_1 \;\&\; (v)[S(e_v) = 0] \Rightarrow S(\langle 2, e\rangle) = 0] \\ \&\; (e)[e \in PRI_1 \;\&\; (Ev)[S(e_v) = 0] \Rightarrow S(\langle 3, e\rangle) = 0].\end{aligned}$$

One proves easily that

$$\mathrm{SAT}(S) \;\&\; \mathrm{SAT}(T) \;\&\; (x)[U(x) = 0 \Leftrightarrow [S(x) = 0 \;\&\; T(x) = 0]] \Rightarrow \mathrm{SAT}(U),$$

and evidently $x \in H \Leftrightarrow (S)[\mathrm{SAT}(S) \Rightarrow S(x)=0]$. We now bring the predicate $\mathrm{SAT}(S) \Rightarrow S(x)=0$ into prenex form and then use the dual of (17.9) and contraction of variables.

REMARK 22. For the last transformation in the proof of Theorem 13 we need the fact that the predicate $S(x)=i$ is absolutely primitive computable. Its characteristic function is given by

$$\begin{aligned}\mathrm{f}(S, x, i) &= 0 \quad \text{if } [S(x) \to 0 \;\&\; i = 0] \vee [S(x) \to 1 \;\&\; i = 1], \\ &= 1 \quad \text{if } [S(x) \to 0 \;\&\; i = 1] \vee [S(x) \to 1 \;\&\; i = 0],\end{aligned}$$

using Lemma 7. We note this rather trivial fact, because the predicate $\mathrm{g}(x) \to i$ is not primitive computable as a predicate of g, x, i when we let g range over all partial multiple valued functions with range in ω—in fact it is not even hyperprojective as we can easily see using Lemma 60.

Put

$$\begin{aligned}S \in \mathrm{WF} \Leftrightarrow &\textit{ there exists no infinite sequence } x_0, x_1, \ldots, x_n, \ldots \\ &\textit{ such that } (n)[S(\langle x_{n+1}, x_n\rangle) = 0].\end{aligned} \tag{17.14}$$

THEOREM 14. (a) *There is an absolutely primitive computable predicate* $R(S,u,v,x)$ *such that*

$$\begin{aligned}x \in H &\Leftrightarrow (ES)[S \in \mathrm{WF} \;\&\; (u)(Ev)R(S, u, v, x)] \\ &\Leftrightarrow (ES)_{HP(x)}[S \in \mathrm{WF} \;\&\; (u)(Ev)R(S, u, v, x)].\end{aligned} \tag{17.15}$$

($HP(x) = HP(D_x)$.)

(b) *Assume that A is semi-hyperprojective from A. There is a predicate $P(S,u,v,x)$, primitive computable from A, such that*

$$
\begin{aligned}
x \in H^A &\Leftrightarrow (ES)[S \in \mathrm{WF} \ \& \ (u)(Ev)P(S, u, v, x)] \\
&\Leftrightarrow (ES)_{HP(A)}[S \in \mathrm{WF} \ \& \ (u)(Ev)P(S, u, v, x)].
\end{aligned} \tag{17.16}
$$

(*The second representation theorem.*)

Proof. Now we analyse the definition of H from the inside, as a transfinite induction. For each S, put

$$
\begin{aligned}
x <_S y &\Leftrightarrow S(\langle x, y\rangle) = 0, \\
D_S &= \{x : (Ey)[S((\langle x, y\rangle) = 0 \vee S(\langle y, x\rangle) = 0]\},
\end{aligned} \tag{17.17}
$$

$$
\begin{aligned}
\mathrm{NICE}(S) \Leftrightarrow & [<_S \textit{ is a partial ordering on } D_S] \\
& \& \ [\langle 1, 0\rangle \textit{ is the only minimal element of } <_S] \\
& \& \ (e)[\langle 2, e\rangle \in D_S \Rightarrow e \in PRI_1 \ \& \ (v)[e_v <_S \langle 2, e\rangle]] \\
& \& \ (e)[\langle 3, e\rangle \in D_S \Rightarrow e \in PRI_1 \ \& \ (Ev)[e_v <_S \langle 3, e\rangle]] \\
& \& \ (x)[x \in D_S \Rightarrow [x = \langle 1, 0\rangle \vee x = \langle 2, (x)_2\rangle \vee x = \langle 3, (x)_2\rangle]].
\end{aligned} \tag{17.18}
$$

We claim that

$$x \in H \Leftrightarrow (ES)[S \in \mathrm{WF} \ \& \ \mathrm{NICE}(S) \ \& \ x \in D_S], \tag{17.19}$$

$$x \in H \Leftrightarrow (ES)_{HP(x)}[S \in \mathrm{WF} \ \& \ \mathrm{NICE}(S) \ \& \ x \in D_S]. \tag{17.20}$$

To prove (17.20) from left to right, put

$$S(u) = 0 \Leftrightarrow |(u)_2| \leqq |x| \ \& \ |(u)_1| < |(u)_2|.$$

To prove (17.19) from right to left, we show by an easy transfinite induction on the partial well-ordering $<_S$ that if $u \in D_S$, then $u \in H$.

Part (a) of the theorem now follows by lower predicate calculus and Lemma 62.

To prove (b) we notice that from part (a) using the hypothesis that A is semi-hyperprojective from A and contraction of variables, we easily construct a predicate $P(S, u, v, x)$, primitive computable from A such that

$$
\begin{aligned}
x \in H^A &\Leftrightarrow (ES)[S \in \mathrm{WF} \ \& \ (u)(Ev)P(S, u, v, x)] \\
&\Leftrightarrow (ES)_{HP(x)}[S \in \mathrm{WF} \ \& \ (u)(Ev)P(S, u, v, x)].
\end{aligned}
$$

Since $HP(x) \subset HP(A)$, we can enlarge the range of the quantifier in the second equivalence to $HP(A)$.

REMARK 23. Theorem 14 implies that if the predicate $S \in \mathrm{WF}$ is hyperprojective, then for a suitable hyperprojective $R(S, x)$, $x \in H \Leftrightarrow (ES)R(S, x)$. This is not always possible, e.g. when $\boldsymbol{\varphi}$ is empty, as we can easily see using Remark 18.

18. **The Spector-Gandy theorem.** In Remark 23 we pointed out that when $\boldsymbol{\varphi} = \varnothing$ then the predicate $S \in \mathrm{WF}$ is not hyperprojective. In [19] we show that with

B and φ as in Example 2, not every predicate of the form $(S)(Eu)(v)R(S, u, v, x)$ with absolutely primitive computable R is semi-hyperprojective. Thus neither of the two representation theorems of §17 yields a normal form theorem for semi-hyperprojective predicates. To obtain such a normal form we prove in this section an abstract version of Spector's theorem on hyperarithmetical quantifiers in [22], proved also by Gandy in [2].

We recall our blanket assumption that φ *consists of total, single-valued functions.*

LEMMA 63. *Assume that A is hyperprojective from A, let $R(S, \boldsymbol{u})$ be semi-hyperprojective from A. Then the predicate*

$$P(\boldsymbol{u}) \Leftrightarrow (ES)_{HP(A)}R(S, \boldsymbol{u}) \tag{18.1}$$

is semi-hyperprojective from A.

Proof. Using the function $g_z(x)$ defined by (17.4), we have

$$(ES)_{HP(A)}R(S, \boldsymbol{u}) \Leftrightarrow (Ez)[(z)_1 \in H^A \;\&\; (z)_2 \in A^* \;\&\; R(g_z, \boldsymbol{u})],$$

from which the lemma follows immediately by (17.3) and the transitivity lemma, Lemma 43.

LEMMA 64. *There is an absolutely primitive computable predicate $R(S, u, v, x, y)$, such that if $y \in H$, then*

$$|x| \leqq |y| \Leftrightarrow (ES)(u)(Ev)R(S, u, v, x, y) \Leftrightarrow (ES)_{HP(x,y)}(u)(Ev)R(S, u, v, x, y) \tag{18.2}$$

(*where $HP(x, y) = HP(D_x \cup D_y)$*).

Proof. Consider the following conditions on a predicate $Q(x, y, z)$:

(a) $x = \langle 1, 0\rangle \;\&\; z = 0$.

(b) $x \neq \langle 1, 0\rangle \;\&\; y = \langle 1, 0\rangle \;\&\; z = 1$.

The remaining conditions all start with the conjunct

$$\begin{aligned}\mathrm{Seq}\,(x) \;\&\; (x)_0 = 2 \;\&\; 2 \leqq (x)_1 \leqq 3 \;\&\; (x)_2 \in PRI_1 \\ \;\&\; \mathrm{Seq}\,(y) \;\&\; (y)_0 = 2 \;\&\; 2 \leqq (y)_1 \leqq 3 \;\&\; (y)_2 \in PRI_1\end{aligned}$$

and then continue with the conditions stated below, where $e = (x)_2$, $m = (y)_2$.

(c) $(x)_1 = 2 \;\&\; (y)_1 = 2 \;\&\; (v)(Eu)Q(e_v, m_u, 0) \;\&\; z = 0$.

(d) $(x)_1 = 2 \;\&\; (y)_1 = 2 \;\&\; (Ev)(u)Q(e_v, m_u, 1) \;\&\; z = 1$.

(e) $(x)_1 = 2 \;\&\; (y)_1 = 3 \;\&\; (v)(u)Q(e_v, m_u, 0) \;\&\; z = 0$.

(f) $(x)_1 = 2 \;\&\; (y)_1 = 3 \;\&\; (Ev)(Eu)Q(e_v, m_u, 1) \;\&\; z = 1$.

(g) $(x)_1 = 3 \;\&\; (y)_1 = 2 \;\&\; (Ev)(Eu)Q(e_v, m_u, 0) \;\&\; z = 0$.

(h) $(x)_1 = 3 \;\&\; (y)_1 = 2 \;\&\; (v)(u)Q(e_v, m_u, 1) \;\&\; z = 1$.

(i) $(x)_1 = 3 \;\&\; (y)_1 = 3 \;\&\; (Ev)(u)Q(e_v, m_u, 0) \;\&\; z = 0$.

(j) $(x)_1 = 3 \;\&\; (y)_1 = 3 \;\&\; (v)(Eu)Q(e_v, m_u, 1) \;\&\; z = 1$.

Let us say that $Q(x, y, z)$ is *nice* if for every x, y, z,

$$Q(x, y, z) \Leftrightarrow (a) \vee (b) \vee \cdots \vee (i) \vee (j). \tag{18.3}$$

We claim that if Q is nice, then

$$x \in H, \quad |x| \leqq |y| \Rightarrow Q(x, y, 0) \,\&\, \bar{Q}(x, y, 1), \tag{18.4}$$

$$|y| < |x| \Rightarrow Q(x, y, 1) \,\&\, \bar{Q}(x, y, 0). \tag{18.5}$$

The proof of (18.4) and (18.5) is by a transfinite induction on *infimum* $(|x|, |y|)$ whose details we omit, since they are similar to those in the proof of Theorem 7.

To complete the proof of the lemma it is enough to find an absolutely primitive computable $R(S, u, v, x, y, z)$ such that the predicates $(ES)(u)(Ev)R(S, u, v, x, y, z)$ and $(ES)_{HP(x,y,z)}(u)(Ev)R(S, u, v, x, y, z)$ are both nice; because then clearly we will have for $y \in H$,

$$|x| \leqq |y| \Leftrightarrow (ES)(u)(Ev)R(S, u, v, x, y, 0), \tag{18.6}$$

$$|x| \leqq |y| \Leftrightarrow (ES)_{HP(x,y)}(u)(Ev)R(S, u, v, x, y, 0). \tag{18.7}$$

For each $f \in B^*$ consider the predicates

$$Q_f(x, y, z) \Leftrightarrow (ES)(u)(Ev)[\{f\}_v(S, u, v, x, y, z) \to 0], \tag{18.8}$$

$$Q'_f(x, y, z) \Leftrightarrow (ES)_{HP(f,x,y,z)}(u)(Ev)[\{f\}_v(S, u, v, x, y, z) \to 0]. \tag{18.9}$$

Let $P_f(x, y, z)$ and $P'_f(x, y, z)$ be defined by the disjunction (a) $\vee$ (b) $\vee \cdots \vee$ (i) $\vee$ (j) when we substitute Q_f and Q'_f respectively for Q. Using the transformations (17.5), (17.6), (17.9) and (17.10) and contraction of variables we can obtain a predicate $Q^*_f(u, v, x, y, z)$ such that

$$P_f(x, y, z) \Leftrightarrow (ES)(u)(Ev)Q^*_f(u, v, x, y, z), \tag{18.10}$$

$$P'_f(x, y, z) \Leftrightarrow (ES)_{HP(f,x,y,z)}(u)(Ev)Q^*_f(u, v, x, y, z). \tag{18.11}$$

Moreover, Q^*_f is a positive propositional combination of absolutely primitive computable predicates and predicates obtained from $\{f\}_v(S, u, v, x, y, z) \to 0$ by the substitution of absolutely primitive computable functions for the variables u, v, x, y, z and the substitution of terms of the form $\lambda t g(S, \boldsymbol{u}, t)$ with g absolutely primitive computable for the variable S. Using the transitivity lemma, Lemma 31, it then follows that there is a combinatorial function $\mathrm{p}(f)$ such that

$$Q^*_f(S, u, v, x, y, z) \Leftrightarrow \{\mathrm{p}(f)\}_v(S, u, v, x, y, z) \to 0. \tag{18.12}$$

We now apply the recursion theorem for search computable functions, Lemma 29, to obtain an $f \in 0^*$ such that

$$\{\mathrm{p}(f)\}_v(S, u, v, x, y, z) = \{f\}_v(S, u, v, x, y, z), \tag{18.13}$$

so that

$$\{f\}_v(S, u, v, x, y, z) \to 0 \Leftrightarrow \{\mathrm{p}(f)\}_v(S, u, v, x, y, z) \to 0. \tag{18.14}$$

If we now apply the prefixes $(ES)(u)(Ev)$ and $(ES)_{HP(x,y,z)}(u)(Ev)$ to both sides of (18.14), we obtain

$$Q_f(x, y, z) \Leftrightarrow P_f(x, y, z), \tag{18.15}$$

$$Q'_f(x, y, z) \Leftrightarrow P'_f(x, y, z) \tag{18.16}$$

so that both Q_f and Q'_f are nice. To obtain a predicate R satisfying (18.6) and (18.7) and hence complete the proof of the lemma it is now enough to apply the normal form theorem, Theorem 1, on $\{f\}_v(S, u, v, x, y, z) \to 0$ and choose an R' so that

$$\{f\}_v(S, u, v, x, y, z) \to 0 \Leftrightarrow (Et)R'(S, u, v, x, y, z, t)$$

and then contract quantifiers once more.

The set H was defined relative to the fixed list of function $\boldsymbol{\varphi}$. Let us write $H(S)$ for the set H defined relative to the list S, $\boldsymbol{\varphi}$. If $x \in H(S)$, put $|x|(S)$ for the ordinal of x in $H(S)$.

LEMMA 64. *There is a combinatorial function* $\mathrm{p}(x)$ *such that*

$$x \in H \Leftrightarrow \mathrm{p}(x) \in H(S), \tag{18.17}$$

$$x \in H \Rightarrow [|x| = |\mathrm{p}(x)|(S)]. \tag{18.18}$$

Proof. It is easy to define a combinatorial function $\mathrm{q}(p, e)$ so that

$$\{\mathrm{q}(p, e)\}_{\mathrm{pr}}((S, \boldsymbol{\varphi})^{\sim}, v) = \{p\}_{\mathrm{pr}}(\{e\}_{\mathrm{pr}}(\boldsymbol{\varphi}^{\sim}, v)); \tag{18.19}$$

put now

$$\begin{aligned} &\mathrm{p}(\langle 1, 0\rangle) = \langle 1, 0\rangle, \quad \mathrm{p}(\langle 2, e\rangle) = \langle 2, \mathrm{q}(p, e)\rangle, \quad \mathrm{p}(\langle 3, e\rangle) = \langle 3, \mathrm{q}(p, e)\rangle, \\ &\mathrm{p}(x) = 0 \quad \text{if } \neg[x = \langle 1, 0\rangle \vee [\mathrm{Seq}\,(x) \,\&\, (x)_0 = 2 \,\&\, 2 \leqq (x)_1 \leqq 3]], \end{aligned} \tag{18.20}$$

using the recursion theorem for primitive computable functions, Lemma 14.

LEMMA 65. *Assume that A is hyperprojective from A. Then the predicate $S \in HP(A)$ is semi-hyperprojective from A, so that for a fixed $x_0 \in A^*$,*

$$S \in HP(A) \Leftrightarrow x_0 \in H(S). \tag{18.21}$$

Proof. To see that $S \in HP(A)$ is semi-hyperprojective from A, we compute:

$$S \in HP(A) \Leftrightarrow (Ez)[(z)_1 \in H^A \,\&\, (z)_2 \in A^* \,\&\, (x)[\mathrm{g}(z, x) \to S(x)]].$$

Now (18.21) follows if we apply part (a) of Theorem 8, relative to the list S, $\boldsymbol{\varphi}$, and with the empty sequence $\boldsymbol{u}$ of individual variables in $P(\boldsymbol{u})$ of (13.6).

THEOREM 15. *Assume that A is hyperprojective from A. There is a predicate $R(S, u, v, x)$, primitive computable from A, such that*

$$x \in H^A \Leftrightarrow (ES)_{HP(A)}(u)(Ev)R(S, u, v, x). \tag{18.22}$$

(*The Spector-Gandy theorem.*)

Proof. Let x_0 be the fixed element of A^* satisfying (18.21). We consider two cases.

Case (a). *supremum* $\{|x_0|(S) : S \in HP(A)\} \geqq \kappa^A$. In this case

$$x \in H^A \Leftrightarrow x \in A^* \,\&\, (ES)_{HP(A)}[|\mathrm{p}(x)|(S) \leqq |x_0|(S)],$$

by Lemma 64. Since A^* is hyperprojective from A, Theorem 8 implies that for a suitable $y_0 \in H^A$ and a combinatorial $\mathrm{q}(x)$,

$$x \in H^A \Leftrightarrow |\mathrm{q}(x)| \leqq |y_0| \;\&\; (ES)_{HP(A)}[|\mathrm{p}(x)|(S) \leqq |x_0|(S)].$$

If we now apply Lemma 63 twice, we get

$$\begin{aligned} x \in H^A \Leftrightarrow (ES)_{HP(A)}(u)(Ev)R(S, u, v, \mathrm{q}(x), y_0) \\ \& \; (ES)_{HP(A)}(ET)_{HP(A)}(u)(Ev)R'(T, S, u, v, \mathrm{p}(x), x_0) \end{aligned}$$

from which the result follows by contractions of the variables.

Case (b). *supremum* $\{|x_0|(S) : S \in HP(A)\} < \kappa^A$. In this case the predicate $S \in HP(A)$ is hyperprojective from A. Put

$$\begin{aligned} \mathrm{f}(S) &= \langle 1, 0\rangle && \textit{if } S \notin HP(A), \\ &= \nu u[u \in H^A \;\&\; S \textit{ is search computable in } H_u], && \textit{if } S \in HP(A). \end{aligned}$$

It follows from Lemma 58 that $\mathrm{f}(S)$ is hyperprojective from A; it is clearly totally defined, and

$$(18.23) \qquad \begin{aligned} &\textit{supremum}_S \; \textit{infimum}_u \{|u| : \mathrm{f}(S) \rightarrow u\} = \kappa^A, \\ &\textit{supremum}_{S \in HP(A)} \; \textit{infimum}_u \{|u| : \mathrm{f}(S) \rightarrow u\} = \kappa^A. \end{aligned}$$

Put

$$(18.24) \qquad \mathrm{f}'(S) \rightarrow u \Leftrightarrow (Eu')[u' \sim u \;\&\; \mathrm{f}(S) \rightarrow u']$$

and notice that $\mathrm{f}'(S)$ also satisfies (18.23) (by Lemma 54) and that the predicate $\mathrm{f}'(S) \rightarrow u$ is semi-hyperprojective from A, since

$$\mathrm{f}'(S) \rightarrow u \Leftrightarrow \mathrm{f}(S) \sim u \Leftrightarrow \chi_{\sim}(\mathrm{f}(S), u) \rightarrow 0.$$

Hence by Theorem 8 there is a combinatorial function $\mathrm{eh}(m, u)$ and a fixed $m \in A^*$ such that

$$\mathrm{f}'(S) \rightarrow u \Leftrightarrow \mathrm{eh}(m, u) \in H(S);$$

if we choose $h' \in 0^*$ such that $\{h'\}_{\mathrm{pr}}((S, \boldsymbol{\varphi})^{\sim}, m, u) = \mathrm{eh}(m, u)$, it follows that

$$(18.25) \qquad (S)[\langle 3, S^1(h', m)\rangle \in H(S)]$$

and that the p.m.v. function

$$(18.26) \qquad \mathrm{f}''(S) \rightarrow u \Leftrightarrow |\mathrm{eh}(m, u)|(S) \leqq |\langle 3, S^1(h', m)\rangle|(S)$$

also satisfies (18.23) since it is total and a subfunction of $\mathrm{f}'(S)$. Hence

$$\begin{aligned} x \in H^A &\Leftrightarrow (ES)(Eu)[|\mathrm{eh}(m, u)|(S) \leqq |\langle 3, S^1(h', m)\rangle|(S) \;\&\; |x| \leqq |u|] \\ &\Leftrightarrow (ES)_{HP(A)}(Eu)[|\mathrm{eh}(m, u)|(S) \leqq |\langle 3, S^1(h', m)\rangle|(S) \;\&\; |x| \leqq |u|] \end{aligned}$$

from which the result follows easily as in Case (a).

REMARK 24. The proof in Case (b) actually shows that for a suitable $R(S, u, v, x)$, primitive computable from A,

$$x \in H^A \Leftrightarrow (ES)(u)(Ev)R(S, u, v, x) \Leftrightarrow (ES)_{HP(A)}(u)(Ev)R(S, u, v, x).$$

When φ is empty such a representation is impossible by Remarks 18 and 23, hence Case (b) does not apply in this case. We do not know of any examples where Case (b) applies.

REMARK 25. For $A=B$, the theorem as stated gives (18.22) with R primitive computable. Using the normal form theorem and contraction of quantifiers one can easily arrange for (18.22) to hold with an absolutely primitive computable R.

REMARK 26. It is an immediate corollary of Theorems 15 and 8 and Lemma 63 that if A is hyperprojective from A, then a predicate $P(\mathbf{x})$ is semi-hyperprojective from A if and only if $P(\mathbf{x})$ is expressible in the form $(ES)_{HP(A)}(u)(Ev)R(S, u, v, \mathbf{x})$, with R primitive computable from A.

19. **A game-theoretic characterization of semi-hyperprojective sets.** The Spector-Gandy theorem gives us a normal form for semi-hyperprojective predicates which however suffers from one defect: it is circular in its use of the restricted quantifier $(ES)_{HP(A)}$. Here we give a normal form which does not involve this circularity and is the abstract analog of the Π_1^1 normal form $(\alpha)(Ey)R(\alpha, x, y)$ for O given by Kleene in [5].

A *game* $\mathscr{G}$ is a quadruple $\{C_0, C_1, T, P(x, y)\}$, where C_0, C_1, T are subsets of B^*, $P(x, y)$ is a predicate on B^* and $C_0 \cap C_1 = \varnothing$. The game $\mathscr{G}$ is played by two players $\mathscr{P}_0$ and $\mathscr{P}_1$. A particular run $\mathscr{G}_x$ of the game $\mathscr{G}$ is determined by an initial "dealing" of an element x of B^*. We put

(19.1) $$x_0 = x;$$

at the $(n+1)$th stage of the game, after the elements $x_0, \ldots, x_n$ have been determined, the element x_{n+1} is determined by the following rules.

(a) If $x_n \in T$, then the game ends and $\mathscr{P}_0$ wins.

(b) If $x_n \notin T$ and $x_n \in C_0$, then $\mathscr{P}_0$ chooses x_{n+1} from the set $\{y : P(x_n, y)\}$; if this set is empty the game ends and $\mathscr{P}_1$ wins.

(c) If $x_n \notin T$ and $x_n \in C_1$, then $\mathscr{P}_1$ chooses x_{n+1} from the set $\{y : P(x_n, y)\}$; if this set is empty the game ends and $\mathscr{P}_1$ wins.

(d) If $x_n \notin T \cup C_0 \cup C_1$, then the game ends and $\mathscr{P}_1$ wins.

Clearly this is a nonsymmetric game in which $\mathscr{P}_0$ attempts to choose (or force $\mathscr{P}_1$ to choose) an element of T by taking advantage of rules (b), (c), while $\mathscr{P}_1$ attempts to block this.

For a fixed game $\mathscr{G}$, put

(19.2) $$E_{\mathscr{G}} = \{x : \mathscr{P}_0 \textit{ has a winning strategy in } \mathscr{G}_x\}.$$

We shall avoid here a formal definition of "strategy" and "winning strategy", since the use we shall make of these terms is very elementary.

For a given game $\mathscr{G}$ we define the set $E'_{\mathscr{G}}$ by the induction:

(19.3)
(a) If $x \in T$, then $x \in E'_{\mathscr{G}}$.
(b) If $x \in C_0$ & $(Ey)[P(x, y)$ & $y \in E'_{\mathscr{G}}]$, then $x \in E'_{\mathscr{G}}$.
(c) If $x \in C_1$ & $(y)[P(x, y) \Rightarrow y \in E'_{\mathscr{G}}]$, then $x \in E'_{\mathscr{G}}$.

LEMMA 66. $E_{\mathscr{G}} = E'_{\mathscr{G}}$.

Proof. We prove first that $E'_{\mathscr{G}} \subset E_{\mathscr{G}}$ by induction on (19.3). If $x \in T$, clearly $x \in E_{\mathscr{G}}$. If $x \in C_0$ and $(Ey)[P(x, y) \ \& \ y \in E'_{\mathscr{G}}]$, then $\mathscr{P}_0$ follows the following strategy: he chooses $x_1 = y$, where $P(x, y) \ \& \ y \in E'_{\mathscr{G}}$ and then follows the strategy for the game $\mathscr{G}_y$ which the induction hypothesis gives him. If $x \in C_1 \ \& \ (y)[P(x,y) \Rightarrow y \in E'_{\mathscr{G}}]$, then no matter which x $\mathscr{P}_1$ chooses, since it must be in $E'_{\mathscr{G}}$, $\mathscr{P}_0$ can follow up with a winning strategy by induction hypothesis.

We prove $E_{\mathscr{G}} \subset E'_{\mathscr{G}}$ by contradiction: assume $x \in E_{\mathscr{G}}$, $x \notin E'_{\mathscr{G}}$. We describe informally how $\mathscr{P}_1$ can play the game so that at each stage n, $x_n \notin E'_{\mathscr{G}}$. Since $T \subset E'_{\mathscr{G}}$, this implies that $\mathscr{P}_0$ cannot win. Assume then that $x_n \notin E'_{\mathscr{G}}$; then $x_n \notin T$. If $x_n \notin C_0 \cup C_1$, we are done, for $\mathscr{P}_1$ has won. If $x_n \in C_0$, then since $x_n \notin E'_{\mathscr{G}}$ no matter what x_{n+1} $\mathscr{P}_0$ chooses we shall have $x_{n+1} \notin E'_{\mathscr{G}}$. If $x_n \in C_1$, then since $x_n \notin E'_{\mathscr{G}}$, $\mathscr{P}_1$ can choose for x_{n+1} an element not in $E'_{\mathscr{G}}$.

We recall our blanket assumption that $\boldsymbol{\varphi}$ consists of single-valued, total functions.

LEMMA 67. *Assume that A is hyperprojective from A. Assume that the game $\mathscr{G}$ is hyperprojective from A, i.e. the sets C_0, C_1, T and the predicate $P(x, y)$ are hyperprojective from A. Then $E_{\mathscr{G}}$ is semi-hyperprojective from A.*

Proof. Let "χ" be a variable for one-place functions, consider the functional $\mathrm{f}(\chi, x)$ defined by

$$\begin{aligned} \mathrm{f}(\chi, x) &= 0 \quad \text{if } x \in T, \\ &= 0 \quad \text{if } x \in C_0 \ \& \ (Ey)[P(x, y) \ \& \ \chi(y) \to 0], \\ &= 0 \quad \text{if } x \in C_1 \ \& \ (y)[P(x, y) \Rightarrow \chi(y) \to 0]. \end{aligned}$$

It is clear that $\mathrm{f}(\chi, x)$ is hyperprojective from A. By the first recursion theorem for hyperprojective functions, Theorem 12, the smallest function $\chi(x)$ satisfying

$$(x)[\mathrm{f}(\chi, x) = \chi(x)] \tag{19.5}$$

is hyperprojective from A. We prove that

$$x \in E_{\mathscr{G}} \Leftrightarrow \chi(x) \to 0 \tag{19.6}$$

by verifying first that the function χ defined by (19.6) satisfies (19.5) and proving then by induction on the definition of $E'_{\mathscr{G}}$ that if $x \in E'_{\mathscr{G}}$ and χ satisfies (19.5), then $\chi(x) \to 0$.

LEMMA 68. *There is a game $\mathscr{G} = \{C_0, C_1, T, P(x, y)\}$ with C_0, C_1, T absolutely primitive computable and $P(x, y)$ in σ_2^0 such that*

$$H = E_{\mathscr{G}}. \tag{19.7}$$

Proof. Put

$$T = \{\langle 1, 0 \rangle\}, \quad C_0 = \{\langle 3, e \rangle : e \in PRI_1\}, \quad C_1 = \{\langle 2, e \rangle : e \in PRI_1\},$$
$$P(x, y) \Leftrightarrow (x)_2 \in PRI_1 \ \& \ (Ev)[y \sim \{(x)_2\}_{\mathrm{pr}}^{\sim}(v)].$$

It is now easy to prove by induction on H that

(19.9) $$\text{if } x \in H, \text{ then } x \in E_{\mathscr{G}}$$

and by induction on (19.3) that

(19.10) $$\text{if } x \in E'_{\mathscr{G}}, \text{ then } x \in H.$$

THEOREM 16. *Assume that A is hyperprojective from A. A predicate $R(\boldsymbol{u})$ is semi-hyperprojective from A if and only if there exists a game $\mathscr{G}=\{C_0, C_1, T, P(x, y)\}$ with absolutely primitive computable C_0, C_1, T and $P(x, y)$ in σ_2^0 and a function* $\mathrm{f}(\boldsymbol{u})$, *primitive computable from A such that*

(19.11) $$R(\boldsymbol{u}) \Leftrightarrow \mathrm{f}(\boldsymbol{u}) \in E_{\mathscr{G}}.$$

Proof is immediate from Lemmas 67 and 68 and Theorem 8.

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